CIS 570 — Introduction to Formal Language Theory — Fall 2008

Exercise Set 2

Model Answers

Exercise 1

First, we put the following text in the file `es2-ex1.sml`:

```
val zero        = Sym.fromString "0";
val one         = Sym.fromString "1";
val zeroPlusOne = StrSet.fromString "0, 1";

(* val mixedEqual : int -> str set

   if n >= 0, then mixedEqual n returns all strings of 0's and 1's
   that have length n and have alphabet {0, 1} *)

fun mixedEqual n =
      StrSet.minus(StrSet.power(zeroPlusOne, n),
                   StrSet.fromList[Str.power([zero], n),
                                   Str.power([one], n)])

(* val mixedLessEqualEven : int -> str set

   if n >= 0, then mixedLessEqualEven n returns all strings of 0's and
   1's that have length <= n, have even length, and have alphabet {0,
   1} *)

fun mixedLessEqualEven n =
      if n = 0
        then mixedEqual 0
      else if n mod 2 = 0
        then StrSet.union(mixedEqual n, mixedLessEqualEven(n - 1))
      else mixedLessEqualEven(n - 1);

(* val mixedLessEqualEvenReg : int -> reg

   if n >= 0, then mixedLessEqualEvenReg n returns a regular
   expression generating all strings of 0's and 1's that have length
   <= n, have even length, and have alphabet {0, 1} *)

fun mixedLessEqualEvenReg n =
      Reg.simplify (NONE, Reg.weakSubset)
                   (Reg.fromStrSet(mixedLessEqualEven n));
```

Next, we start Forlan, and then proceed as follows:

```
- use "es2-ex1.sml";
[opening es2-ex1.sml]
val zero = - : sym
val one = - : sym
val zeroPlusOne = - : str set
val mixedEqual = fn : int -> str set
val mixedLessEqualEven = fn : int -> str set
val mixedLessEqualEvenReg = fn : int -> reg
val it = () : unit
- val reg = mixedLessEqualEvenReg 4;
val reg = - : reg
- Reg.size reg;
val it = 51 : int
- Reg.output("", reg);
0(0(01 + 1(0 + 1)) + 1(% + (0 + 1)(0 + 1))) +
1(0(% + (0 + 1)(0 + 1)) + 1(0(0 + 1) + 10))
val it = () : unit
```

This took about 8 minutes on a moderately fast laptop.

Exercise 2

(a) Suppose $A, B \in \mathbf{Lan}$. We must show that $(A^*B)^*A^* = (A \cup B)^*$. We begin by proving three lemmas.

Lemma ES2.2.1
$(A^*B)^*A^* \subseteq (A \cup B)^*$.

Proof. Since $A \subseteq A \cup B$, we have that $A^* \subseteq (A \cup B)^*$. Furthermore, $B \subseteq A \cup B \subseteq (A \cup B)^*$. Thus, by Proposition 3.2.9(4), it follows that $A^*B \subseteq (A \cup B)^*(A \cup B)^* = (A \cup B)^*$. Then, by Proposition 3.2.9(5), we have that $(A^*B)^* \subseteq ((A \cup B)^*)^* = (A \cup B)^*$. Finally, by Proposition 3.2.9(4), it follows that $(A^*B)^*A^* \subseteq (A \cup B)^*(A \cup B)^* = (A \cup B)^*$. $\square$

Lemma ES2.2.2
$(A \cup B)(A^*B)^*A^* \subseteq (A^*B)^*A^*$.

Proof. Suppose $w \in (A \cup B)(A^*B)^*A^*$. We must show that $w \in (A^*B)^*A^*$. By the assumption concerning w, we have that $w = xyz$, for some $x \in A \cup B$, $y \in (A^*B)^*$ and $z \in A^*$. There are two main cases to consider.

- $(x \in A)$ Since $y \in (A^*B)^*$, we have that $y \in (A^*B)^n$, for some $n \in \mathbb{N}$. There are two subcases to consider.
  - $(n = 0)$ Then $y \in (A^*B)^0 = \{\%\}$, so that $y = \%$. Hence $w = xyz = x\%z = xz \in AA^* \subseteq A^*$, so that $w = \%w \in (A^*B)^*A^*$.
  - $(n > 0)$ Then $y \in (A^*B)^n = A^*B(A^*B)^{n-1}$, so that $y = tuv$, for some $t \in A^*$, $u \in B$ and $v \in (A^*B)^{n-1}$. Hence $w = xyz = xtuvz$. Since $xt \in AA^* \subseteq A^*$, we have that $(xt)u \in A^*B$. Then $(xtu)v \in (A^*B)(A^*B)^{n-1} = (A^*B)^n \subseteq (A^*B)^*$. Finally, we have that $w = (xtuv)z \in (A^*B)^*A^*$.

- $(x \in B)$ Then $x = \%x \in A^*B$, so that $xy \in (A^*B)(A^*B)^* \subseteq (A^*B)^*$. Hence $w = (xy)z \in (A^*B)^*A^*$.

□

Lemma ES2.2.3
$(A \cup B)^* \subseteq (A^*B)^*A^*$.

Proof. It will suffice to show that, for all $n \in \mathbb{N}$, $(A \cup B)^n \subseteq (A^*B)^*A^*$. We proceed by mathematical induction.

(Basis Step) We have that $(A \cup B)^0 = \{\%\} = \{\%\}\{\%\} \subseteq (A^*B)^*A^*$.

(Inductive Step) Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: $(A \cup B)^n \subseteq (A^*B)^*A^*$. Then

$$\begin{aligned}(A \cup B)^{n+1} &= (A \cup B)(A \cup B)^n \\ &\subseteq (A \cup B)(A^*B)^*A^* && \text{(inductive hypothesis)} \\ &\subseteq (A^*B)^*A^* && \text{(Lemma ES2.2.2).}\end{aligned}$$

□

By Lemmas ES2.2.1 and ES2.2.3, we have that $(A^*B)^*A^* \subseteq (A \cup B)^* \subseteq (A^*B)^*A^*$. Thus $(A^*B)^*A^* = (A \cup B)^*$.

(b) Suppose $\alpha, \beta \in \mathbf{Reg}$. Then

$$\begin{aligned}L((\alpha^*\beta)^*\alpha^*) &= L((\alpha^*\beta)^*)L(\alpha^*) \\ &= L(\alpha^*\beta)^*L(\alpha)^* \\ &= (L(\alpha^*)L(\beta))^*L(\alpha)^* \\ &= (L(\alpha)^*L(\beta))^*L(\alpha)^* \\ &= (L(\alpha) \cup L(\beta))^* && \text{(Part (a))} \\ &= L(\alpha + \beta)^* \\ &= L((\alpha + \beta)^*),\end{aligned}$$

so that $(\alpha^*\beta)^*\alpha^* \approx (\alpha + \beta)^*$.

Exercise 3

(a)

$$(\mathsf{10} + (\mathsf{0} + \mathsf{11})(\mathsf{01})^*(\mathsf{1} + \mathsf{00}))^*$$

(b) Define languages A and B by:

$$\begin{aligned}A &= \{0, 11\}\{01\}^*\{1, 00\}, \\ B &= \{10\} \cup A.\end{aligned}$$

Because $\% \notin A$ and $\% \neq \mathsf{10}$, we have that $\% \notin B$.

Then

$$\begin{aligned} L(\alpha) &= L((\mathsf{10} + (\mathsf{0} + \mathsf{11})(\mathsf{01})^*(\mathsf{1} + \mathsf{00}))^*) \\ &= L(\mathsf{10} + (\mathsf{0} + \mathsf{11})(\mathsf{01})^*(\mathsf{1} + \mathsf{00}))^* \\ &= (\{\mathsf{10}\} \cup L((0 + 11)(01)^*(1 + 00)))^* \\ &= (\{\mathsf{10}\} \cup A)^* \\ &= B^*. \end{aligned}$$

Hence, it will suffice to show that $B^* = X$.

Define languages Y and Z by:

$$\begin{aligned} Y &= \{\, w \in \{\mathsf{0}, \mathsf{1}\}^* \mid \mathbf{diff}\, w = 3m + 1, \text{ for some } m \in \mathbb{Z} \,\}, \\ Z &= \{\, w \in \{\mathsf{0}, \mathsf{1}\}^* \mid \mathbf{diff}\, w = 3m + 2, \text{ for some } m \in \mathbb{Z} \,\}. \end{aligned}$$

Lemma ES2.3.1
For all $w \in \{\mathsf{0}, \mathsf{1}\}^*$, w is an element of exactly one of X, Y and Z.

Proof. Suppose $w \in \{\mathsf{0}, \mathsf{1}\}^*$. There are unique $n \in \mathbb{Z}$ and $m \in [0:2]$ such that $\mathbf{diff}\, w = 3n + m$. Thus w is in exactly one of X, Y and Z. □

Lemma ES2.3.2

(1) $\%, \mathsf{01}, \mathsf{10} \in X$.

(2) $\mathsf{1}, \mathsf{00} \in Y$.

(3) $\mathsf{0}, \mathsf{11} \in Z$.

(4) $XX \subseteq X$.

(5) $X^* \subseteq X$.

(6) $\{\mathsf{01}\}^* \subseteq X$.

(7) $ZX \subseteq Z$.

(8) $YZ \subseteq X$.

(9) $ZY \subseteq X$.

(10) $ZZ \subseteq Y$.

(11) $\% \notin Y$ *and* $\% \notin Z$.

Proof.

(1) Because $\mathbf{diff}\, \% = 0 = 3 \cdot 0$, $\mathbf{diff}(\mathsf{01}) = 0 = 3 \cdot 0$ and $\mathbf{diff}(\mathsf{10}) = 0 = 3 \cdot 0$, we have that $\%, \mathsf{01}, \mathsf{10} \in X$.

(2) Because $\mathbf{diff}\, \mathsf{1} = 1 = 3 \cdot 0 + 1$ and $\mathbf{diff}(\mathsf{00}) = -2 = 3 \cdot -1 + 1$, we have that $\mathsf{1}, \mathsf{00} \in Y$.

(3) Because $\mathbf{diff}\,\mathsf{0} = -1 = 3 \cdot -1 + 2$ and $\mathbf{diff}(\mathsf{11}) = 2 = 3 \cdot 0 + 2$, we have that $\mathsf{0}, \mathsf{11} \in Z$.

(4) Suppose $w \in XX$. Thus $w = xy$ for some $x, y \in X$. Thus $\mathbf{diff}\,x = 3n$ for some $n \in \mathbb{N}$, and $\mathbf{diff}\,y = 3m$ for some $m \in \mathbb{N}$. Hence $\mathbf{diff}(xy) = \mathbf{diff}\,x + \mathbf{diff}\,y = 3n + 3m = 3(n + m)$ and $n + m \in \mathbb{Z}$, showing that $w = xy \in X$.

(5) It will suffice to show that, for all $n \in \mathbb{N}$, $X^n \subseteq X$. We proceed by mathematical induction.

- (basis step) By Part (1), we have that $X^0 = \{\%\} \subseteq X$.
- (inductive step) Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: $X^n \subseteq X$. Then $X^{n+1} = XX^n \subseteq XX \subseteq X$, by the inductive hypothesis and Part (4).

(6) By Part (1), we have that $\{\mathsf{01}\} \subseteq X$. Hence $\{\mathsf{01}\}^* \subseteq X^* \subseteq X$, by Part (5).

(7) Suppose $w \in ZX$. Thus $w = xy$ for some $x \in Z$ and $y \in X$. Thus $\mathbf{diff}\,x = 3n + 2$ for some $n \in \mathbb{N}$, and $\mathbf{diff}\,y = 3m$ for some $m \in \mathbb{N}$. Hence $\mathbf{diff}(xy) = \mathbf{diff}\,x + \mathbf{diff}\,y = 3n + 2 + 3m = 3(n + m) + 2$ and $n + m \in \mathbb{Z}$, showing that $w = xy \in Z$.

(8) Suppose $w \in YZ$. Thus $w = xy$ for some $x \in Y$ and $y \in Z$. Thus $\mathbf{diff}\,x = 3n + 1$ for some $n \in \mathbb{N}$, and $\mathbf{diff}\,y = 3m + 2$ for some $m \in \mathbb{N}$. Hence $\mathbf{diff}(xy) = \mathbf{diff}\,x + \mathbf{diff}\,y = 3n + 1 + 3m + 2 = 3(n + m) + 3 = 3(n + m + 1)$ and $n + m + 1 \in \mathbb{Z}$, showing that $w = xy \in X$.

(9) Suppose $w \in ZY$. Thus $w = xy$ for some $x \in Z$ and $y \in Y$. Thus $\mathbf{diff}\,x = 3n + 2$ for some $n \in \mathbb{N}$, and $\mathbf{diff}\,y = 3m + 1$ for some $m \in \mathbb{N}$. Hence $\mathbf{diff}(xy) = \mathbf{diff}\,x + \mathbf{diff}\,y = 3n + 2 + 3m + 1 = 3(n + m) + 3 = 3(n + m + 1)$ and $n + m + 1 \in \mathbb{Z}$, showing that $w = xy \in X$.

(10) Suppose $w \in ZZ$. Thus $w = xy$ for some $x \in Z$ and $y \in Z$. Thus $\mathbf{diff}\,x = 3n + 2$ for some $n \in \mathbb{N}$, and $\mathbf{diff}\,y = 3m + 2$ for some $m \in \mathbb{N}$. Hence $\mathbf{diff}(xy) = \mathbf{diff}\,x + \mathbf{diff}\,y = 3n + 2 + 3m + 2 = 3(n + m) + 3 + 1 = 3(n + m + 1) + 1$ and $n + m + 1 \in \mathbb{Z}$, showing that $w = xy \in Y$.

(11) By Part (1), we have that $\% \in X$. Thus, by Lemma ES2.3.1, we have that $\% \notin Y$ and $\% \notin Z$.

□

Lemma ES2.3.3
$B^* \subseteq X$.

Proof. By Lemma ES2.3.2(3), we have that $\{\mathsf{0}, \mathsf{11}\} \subseteq Z$, and by Lemma ES2.3.2(6), it follows that $\{\mathsf{01}\}^* \subseteq X$. Hence we have that $\{\mathsf{0}, \mathsf{11}\}\{\mathsf{01}\}^* \subseteq ZX \subseteq Z$, by Lemma ES2.3.2(7). By Lemma ES2.3.2(2), we have that $\{\mathsf{1}, \mathsf{00}\} \subseteq Y$. Thus $A = \{\mathsf{0}, \mathsf{11}\}\{\mathsf{01}\}^*\{\mathsf{1}, \mathsf{00}\} \subseteq ZY \subseteq X$, by Lemma ES2.3.2(9). By Lemma ES2.3.2(1), we have that $\{\mathsf{10}\} \subseteq X$. Thus $B = \{\mathsf{10}\} \cup A \subseteq X \cup X = X$, so that $B^* \subseteq X^* \subseteq X$, by Lemma ES2.3.2(5). □

Lemma ES2.3.4

(1) For all $x, y \in \{\mathsf{0}, \mathsf{1}\}^*$, if $xy \in X$ and $x \in X$, then $y \in X$.

(2) For all $x, y \in \{\mathsf{0}, \mathsf{1}\}^*$, if $xy \in Y$ and $x \in X$, then $y \in Y$.

(3) For all $x, y \in \{\mathsf{0}, \mathsf{1}\}^*$, if $xy \in Y$ and $x \in Y$, then $y \in X$.

(4) *For all* $x, y \in \{0,1\}^*$, *if* $xy \in Z$ *and* $x \in Z$, *then* $y \in X$.

(5) *For all* $x, y \in \{0,1\}^*$, *if* $xy \in Z$ *and* $x \in Y$, *then* $y \in Y$.

(6) *For all* $x, y \in \{0,1\}^*$, *if* $xy \in Y$ *and* $x \in Z$, *then* $y \in Z$.

(7) *For all* $x, y \in \{0,1\}^*$, *if* $xy \in X$ *and* $x \in Y$, *then* $y \in Z$.

(8) *For all* $x, y \in \{0,1\}^*$, *if* $xy \in X$ *and* $x \in Z$, *then* $y \in Y$.

Proof.

(1) Suppose $x, y \in \{0,1\}^*$, $xy \in X$ and $x \in X$. Thus $\textbf{diff}\, x + \textbf{diff}\, y = \textbf{diff}(xy) = 3n$, for some $n \in \mathbb{Z}$, and $\textbf{diff}\, x = 3m$, for some $m \in \mathbb{Z}$. Hence $3m + \textbf{diff}\, y = \textbf{diff}\, x + \textbf{diff}\, y = 3n$, so that $\textbf{diff}\, y = 3n - 3m = 3(n-m)$ and $n - m \in \mathbb{Z}$. Thus $y \in X$.

(2) Suppose $x, y \in \{0,1\}^*$, $xy \in Y$ and $x \in X$. Thus $\textbf{diff}\, x + \textbf{diff}\, y = \textbf{diff}(xy) = 3n+1$, for some $n \in \mathbb{Z}$, and $\textbf{diff}\, x = 3m$, for some $m \in \mathbb{Z}$. Hence $3m + \textbf{diff}\, y = \textbf{diff}\, x + \textbf{diff}\, y = 3n+1$, so that $\textbf{diff}\, y = 3n + 1 - 3m = 3(n-m) + 1$ and $n - m \in \mathbb{Z}$. Thus $y \in Y$.

(3) Suppose $x, y \in \{0,1\}^*$, $xy \in Y$ and $x \in Y$. Thus $\textbf{diff}\, x + \textbf{diff}\, y = \textbf{diff}(xy) = 3n+1$, for some $n \in \mathbb{Z}$, and $\textbf{diff}\, x = 3m+1$, for some $m \in \mathbb{Z}$. Hence $3m+1+\textbf{diff}\, y = \textbf{diff}\, x + \textbf{diff}\, y = 3n+1$, so that $\textbf{diff}\, y = 3n + 1 - 3m - 1 = 3(n-m)$ and $n - m \in \mathbb{Z}$. Thus $y \in X$.

(4) Suppose $x, y \in \{0,1\}^*$, $xy \in Z$ and $x \in Z$. Thus $\textbf{diff}\, x + \textbf{diff}\, y = \textbf{diff}(xy) = 3n+2$, for some $n \in \mathbb{Z}$, and $\textbf{diff}\, x = 3m+2$, for some $m \in \mathbb{Z}$. Hence $3m+2+\textbf{diff}\, y = \textbf{diff}\, x + \textbf{diff}\, y = 3n+2$, so that $\textbf{diff}\, y = 3n + 2 - 3m - 2 = 3(n-m)$ and $n - m \in \mathbb{Z}$. Thus $y \in X$.

(5) Suppose $x, y \in \{0,1\}^*$, $xy \in Z$ and $x \in Y$. Thus $\textbf{diff}\, x + \textbf{diff}\, y = \textbf{diff}(xy) = 3n+2$, for some $n \in \mathbb{Z}$, and $\textbf{diff}\, x = 3m+1$, for some $m \in \mathbb{Z}$. Hence $3m+1+\textbf{diff}\, y = \textbf{diff}\, x + \textbf{diff}\, y = 3n+2$, so that $\textbf{diff}\, y = 3n + 2 - 3m - 1 = 3(n-m) + 1$ and $n - m \in \mathbb{Z}$. Thus $y \in Y$.

(6) Suppose $x, y \in \{0,1\}^*$, $xy \in Y$ and $x \in Z$. Thus $\textbf{diff}\, x + \textbf{diff}\, y = \textbf{diff}(xy) = 3n+1$, for some $n \in \mathbb{Z}$, and $\textbf{diff}\, x = 3m+2$, for some $m \in \mathbb{Z}$. Hence $3m+2+\textbf{diff}\, y = \textbf{diff}\, x + \textbf{diff}\, y = 3n+1$, so that $\textbf{diff}\, y = 3n+1-3m-2 = 3(n-m)+-1 = 3(n-m)+(-3+2) = 3(n-m)+3\cdot-1+2 = 3(n-m-1)+2$ and $n - m - 1 \in \mathbb{Z}$. Thus $y \in Z$.

(7) Suppose $x, y \in \{0,1\}^*$, $xy \in X$ and $x \in Y$. Thus $\textbf{diff}\, x + \textbf{diff}\, y = \textbf{diff}(xy) = 3n$, for some $n \in \mathbb{Z}$, and $\textbf{diff}\, x = 3m+1$, for some $m \in \mathbb{Z}$. Hence $3m+1+\textbf{diff}\, y = \textbf{diff}\, x + \textbf{diff}\, y = 3n$, so that $\textbf{diff}\, y = 3n - 3m - 1 = 3(n-m) + -1 = 3(n-m) + (-3+2) = 3(n-m) + 3\cdot -1 + 2 = 3(n-m-1)+2$ and $n - m - 1 \in \mathbb{Z}$. Thus $y \in Z$.

(8) Suppose $x, y \in \{0,1\}^*$, $xy \in X$ and $x \in Z$. Thus $\textbf{diff}\, x + \textbf{diff}\, y = \textbf{diff}(xy) = 3n$, for some $n \in \mathbb{Z}$, and $\textbf{diff}\, x = 3m+2$, for some $m \in \mathbb{Z}$. Hence $3m+2+\textbf{diff}\, y = \textbf{diff}\, x + \textbf{diff}\, y = 3n$, so that $\textbf{diff}\, y = 3n - 3m - 2 = 3(n-m) + -2 = 3(n-m) + (-3+1) = 3(n-m) + 3\cdot -1 + 1 = 3(n-m-1)+1$ and $n - m - 1 \in \mathbb{Z}$. Thus $y \in Y$.

□

Lemma ES2.3.5
$Y \subseteq \{01\}^*\{1, 00\}X$.

Proof. Suppose $x \in Y$. We must show that $x \in \{\mathtt{01}\}^*\{\mathtt{1},\mathtt{00}\}X$. Let y be the longest prefix of x that is in $\{\mathtt{01}\}^*$. (y is well-defined, as it could always be %.) Let $z \in \{\mathtt{0},\mathtt{1}\}^*$ be such that $x = yz$. By Lemma ES2.3.2(6), we have that $y \in \{\mathtt{01}\}^* \subseteq X$. Because $yz = x \in Y$ and $y \in X$, Lemma ES2.3.4(2) tells us that $z \in Y$. Thus, by Lemma ES2.3.2(11), $z \neq \%$, so that $z = au$ for some $a \in \{\mathtt{0},\mathtt{1}\}$ and $u \in \{\mathtt{0},\mathtt{1}\}^*$. Because $a \in \{\mathtt{0},\mathtt{1}\}$, there are two subcases to consider.

- Suppose $a = \mathtt{0}$. Thus $z = au = \mathtt{0}u$ and $x = yz = y\mathtt{0}u$. By Lemma ES2.3.2(3), we have that $\mathtt{0} \in Z$. Because $\mathtt{0}u = z \in Y$ and $\mathtt{0} \in Z$, Lemma ES2.3.4(6) tells us that $u \in Z$. Hence $u \neq \%$, by Lemma ES2.3.2(11), so that $u = bv$ for some $b \in \{\mathtt{0},\mathtt{1}\}$ and $v \in \{\mathtt{0},\mathtt{1}\}^*$. Thus $x = y\mathtt{0}bv$. By the definition of y, we have that $b \neq \mathtt{1}$, so that $b = \mathtt{0}$. Thus $u = \mathtt{0}v$ and $x = y\mathtt{00}v$. Because $\mathtt{0}v = u \in Z$ and $\mathtt{0} \in Z$, Lemma ES2.3.4(4) tells us that $v \in X$. Thus $x = y\mathtt{00}v \in \{\mathtt{01}\}^*\{\mathtt{1},\mathtt{00}\}X$.

- Suppose $a = \mathtt{1}$. Thus $z = au = \mathtt{1}u$ and $x = yz = y\mathtt{1}u$. By Lemma ES2.3.2(2), we have that $\mathtt{1} \in Y$. Because $\mathtt{1}u = z \in Y$ and $\mathtt{1} \in Y$, Lemma ES2.3.4(3) tells us that $u \in X$. Thus $x = y\mathtt{1}u \in \{\mathtt{01}\}^*\{\mathtt{1},\mathtt{00}\}X$.

□

Lemma ES2.3.6
$X \subseteq B^*$.

Proof. Since $X \subseteq \{\mathtt{0},\mathtt{1}\}^*$, it will suffice to show that, for all $w \in \{\mathtt{0},\mathtt{1}\}^*$,

$$\text{if } w \in X, \text{ then } w \in B^*.$$

Suppose $w \in \{\mathtt{0},\mathtt{1}\}^*$, and assume the inductive hypothesis: for all $x \in \{\mathtt{0},\mathtt{1}\}^*$, if x is a proper substring of w, then

$$\text{if } x \in X, \text{ then } x \in B^*.$$

We must show that

$$\text{if } w \in X, \text{ then } w \in B^*.$$

Suppose $w \in X$. We must show that $w \in B^*$. There are three cases to consider.

- Suppose $w = \%$. Then $w = \% \in B^*$.

- Suppose $w = \mathtt{0}x$ for some $x \in \{\mathtt{0},\mathtt{1}\}^*$. By Lemma ES2.3.2(3), we have that $\mathtt{0} \in Z$. Because $\mathtt{0}x = w \in X$ and $\mathtt{0} \in Z$, Lemma ES2.3.4(8) tells us that $x \in Y$. Hence, by Lemma ES2.3.5, we have that $x \in Y \subseteq \{\mathtt{01}\}^*\{\mathtt{1},\mathtt{00}\}X$. Thus $w = \mathtt{0}x \in \{\mathtt{0},\mathtt{11}\}\{\mathtt{01}\}^*\{\mathtt{1},\mathtt{00}\}X = AX \subseteq BX$, so that $w = yz$ for some $y \in B$ and $z \in X$. Because $y \in B$, we have that $y \neq \%$. Thus z is a proper substring of w, so that the inductive hypothesis tells us that $z \in B^*$. Hence $w = yz \in BB^* \subseteq B^*$.

- Suppose $w = \mathtt{1}x$ for some $x \in \{\mathtt{0},\mathtt{1}\}^*$. By Lemma ES2.3.2(2), we have that $\mathtt{1} \in Y$. Because $\mathtt{1}x = w \in X$ and $\mathtt{1} \in Y$, we have that $x \in Z$, by Lemma ES2.3.4(7). Thus, by Lemma ES2.3.2(11), $x \neq \%$, so that $x = ay$ for some $a \in \{\mathtt{0},\mathtt{1}\}$ and $y \in \{\mathtt{0},\mathtt{1}\}^*$. Because $a \in \{\mathtt{0},\mathtt{1}\}$, there are two subcases to consider.

- Suppose $a = \mathsf{0}$. Thus $x = \mathsf{0}y$ and $w = \mathsf{1}x = \mathsf{10}y$. We have that $\mathsf{10} \in B$. By Lemma ES2.3.2(3), we have that $\mathsf{0} \in Z$. Because $\mathsf{0}y = x \in Z$ and $\mathsf{0} \in Z$, Lemma ES2.3.4(4) tells us that $y \in X$. But y is a proper substring of w, and thus the inductive hypothesis tells us that $y \in B^*$. Thus $w = \mathsf{10}y \in BB^* \subseteq B^*$.
- Suppose $a = \mathsf{1}$. Thus $x = \mathsf{1}y$ and $w = \mathsf{1}x = \mathsf{11}y$. By Lemma ES2.3.2(2), we have that $\mathsf{1} \in Y$. Because $\mathsf{1}y = x \in Z$ and $\mathsf{1} \in Y$, Lemma ES2.3.4(5) tells us that $y \in Y$. Hence, by Lemma ES2.3.5, we have that $y \in Y \subseteq \{\mathsf{01}\}^*\{\mathsf{1}, \mathsf{00}\}X$. Thus $w = \mathsf{11}y \in \{\mathsf{0}, \mathsf{11}\}\{\mathsf{01}\}^*\{\mathsf{1}, \mathsf{00}\}X = AX \subseteq BX$, so that $w = uv$ for some $u \in B$ and $v \in X$. Because $u \in B$, we have that $u \neq \%$. Thus v is a proper substring of w, so that the inductive hypothesis tells us that $v \in B^*$. Hence $w = uv \in BB^* \subseteq B^*$.

□

By Propositions ES2.3.3 and ES2.3.6, we have that $B^* \subseteq X \subseteq B^*$, so that $B^* = X$.