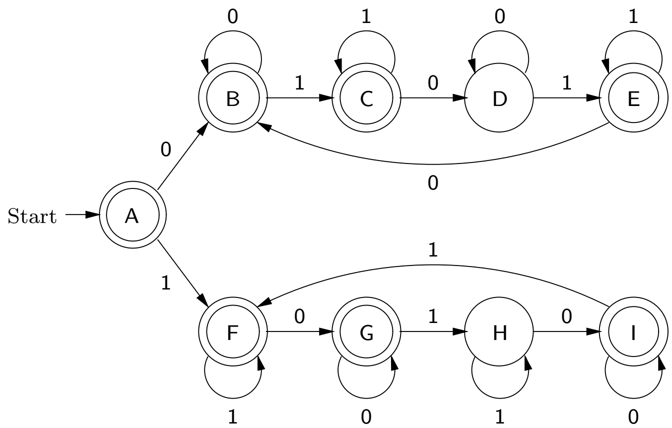
CIS 570 — Introduction to Formal Language Theory — Fall 2008

Exercise Set 4

Model Answers

Exercise 1

(a)



(b) We have that, for all $w \in \{0, 1\}^*$:

- $w \in X$ iff **zo** w is even or **oz** w is even; and
- $w \notin X$ iff **zo** w is odd and **oz** w is odd.

Because **alphabet** $M = \{0, 1\}$, we have that $\Lambda_{M,q} \subseteq \{0, 1\}^*$ for all $q \in Q_M$.

Lemma ES4.1.1

(A) For all $w \in \Lambda_{\mathsf{A}}$, $w = \%$.

(B) For all $w \in \Lambda_{\mathsf{B}}$, **zo** w *is even,* **oz** w *is even and* 0 *is a suffix of* w.

(C) For all $w \in \Lambda_{\mathsf{C}}$, **zo** w *is odd,* **oz** w *is even and* 1 *is a suffix of* w.

(D) For all $w \in \Lambda_{\mathsf{D}}$, **zo** w *is odd,* **oz** w *is odd and* 0 *is a suffix of* w.

(E) For all $w \in \Lambda_{\mathsf{E}}$, **zo** w *is even,* **oz** w *is odd and* 1 *is a suffix of* w.

(F) For all $w \in \Lambda_{\mathsf{F}}$, **zo** w *is even,* **oz** w *is even and* 1 *is a suffix of* w.

(G) For all $w \in \Lambda_{\mathsf{G}}$, **zo** w *is even,* **oz** w *is odd and* 0 *is a suffix of* w.

(H) For all $w \in \Lambda_{\mathsf{H}}$, **zo** w *is odd,* **oz** w *is odd and* 1 *is a suffix of* w.

(I) *For all* $w \in \Lambda_{\mathsf{I}}$, $\mathbf{zo}\, w$ *is odd,* $\mathbf{oz}\, w$ *is even and* $\mathsf{0}$ *is a suffix of* w.

Proof. We proceed by induction on Λ. There are 19 parts to show.

- (empty string) We have that $\% = \%$.

- $(\mathsf{A}, \mathsf{0} \to \mathsf{B})$ Suppose $w \in \Lambda_{\mathsf{A}}$ (so that $w \in \{\mathsf{0}, \mathsf{1}\}^*$) and assume the inductive hypothesis, $w = \%$. We have that $\mathbf{zo}(w\mathsf{0}) = \mathbf{zo}\, w = \mathbf{zo}\, \% = 0$ is even, $\mathbf{oz}(w\mathsf{0}) = 0$ is even, and $\mathsf{0}$ is a suffix of $w\mathsf{0}$.

- $(\mathsf{A}, \mathsf{1} \to \mathsf{F})$ Suppose $w \in \Lambda_{\mathsf{A}}$ and assume the inductive hypothesis, $w = \%$. We have that $\mathbf{zo}(w\mathsf{1}) = 0$ is even, $\mathbf{oz}(w\mathsf{1}) = \mathbf{oz}\, w = \mathbf{oz}\, \% = 0$ is even, and $\mathsf{1}$ is a suffix of $w\mathsf{1}$.

- $(\mathsf{B}, \mathsf{0} \to \mathsf{B})$ Suppose $w \in \Lambda_{\mathsf{B}}$ (so that $w \in \{\mathsf{0}, \mathsf{1}\}^*$) and assume the inductive hypothesis, $\mathbf{zo}\, w$ is even, $\mathbf{oz}\, w$ is even and $\mathsf{0}$ is a suffix of w. We have that $\mathbf{zo}(w\mathsf{0}) = \mathbf{zo}\, w$ is even, $\mathbf{oz}(w\mathsf{0}) = \mathbf{oz}\, w$ is even, and $\mathsf{0}$ is a suffix of $w\mathsf{0}$.

- $(\mathsf{B}, \mathsf{1} \to \mathsf{C})$ Suppose $w \in \Lambda_{\mathsf{B}}$ and assume the inductive hypothesis, $\mathbf{zo}\, w$ is even, $\mathbf{oz}\, w$ is even and $\mathsf{0}$ is a suffix of w. We have that $\mathbf{zo}(w\mathsf{1}) = \mathbf{zo}\, w + 1$ is odd, $\mathbf{oz}(w\mathsf{1}) = \mathbf{oz}\, w$ is even, and $\mathsf{1}$ is a suffix of $w\mathsf{1}$.

- $(\mathsf{C}, \mathsf{0} \to \mathsf{D})$ Suppose $w \in \Lambda_{\mathsf{C}}$ and assume the inductive hypothesis, $\mathbf{zo}\, w$ is odd, $\mathbf{oz}\, w$ is even and $\mathsf{1}$ is a suffix of w. We have that $\mathbf{zo}(w\mathsf{0}) = \mathbf{zo}\, w$ is odd, $\mathbf{oz}(w\mathsf{0}) = \mathbf{oz}\, w + 1$ is odd, and $\mathsf{0}$ is a suffix of $w\mathsf{0}$.

- $(\mathsf{C}, \mathsf{1} \to \mathsf{C})$ Suppose $w \in \Lambda_{\mathsf{C}}$ and assume the inductive hypothesis, $\mathbf{zo}\, w$ is odd, $\mathbf{oz}\, w$ is even and $\mathsf{1}$ is a suffix of w. We have that $\mathbf{zo}(w\mathsf{1}) = \mathbf{zo}\, w$ is odd, $\mathbf{oz}(w\mathsf{1}) = \mathbf{oz}\, w$ is even, and $\mathsf{1}$ is a suffix of $w\mathsf{1}$.

- $(\mathsf{D}, \mathsf{0} \to \mathsf{D})$ Suppose $w \in \Lambda_{\mathsf{D}}$ and assume the inductive hypothesis, $\mathbf{zo}\, w$ is odd, $\mathbf{oz}\, w$ is odd and $\mathsf{0}$ is a suffix of w. We have that $\mathbf{zo}(w\mathsf{0}) = \mathbf{zo}\, w$ is odd, $\mathbf{oz}(w\mathsf{0}) = \mathbf{oz}\, w$ is odd, and $\mathsf{0}$ is a suffix of $w\mathsf{0}$.

- $(\mathsf{D}, \mathsf{1} \to \mathsf{E})$ Suppose $w \in \Lambda_{\mathsf{D}}$ and assume the inductive hypothesis, $\mathbf{zo}\, w$ is odd, $\mathbf{oz}\, w$ is odd and $\mathsf{0}$ is a suffix of w. We have that $\mathbf{zo}(w\mathsf{1}) = \mathbf{zo}\, w + 1$ is even, and $\mathbf{oz}(w\mathsf{1}) = \mathbf{oz}\, w$ is odd, and $\mathsf{1}$ is a suffix of $w\mathsf{1}$.

- $(\mathsf{E}, \mathsf{0} \to \mathsf{B})$ Suppose $w \in \Lambda_{\mathsf{E}}$ and assume the inductive hypothesis, $\mathbf{zo}\, w$ is even, $\mathbf{oz}\, w$ is odd and $\mathsf{1}$ is a suffix of w. We have that $\mathbf{zo}(w\mathsf{0}) = \mathbf{zo}\, w$ is even, $\mathbf{oz}(w\mathsf{0}) = \mathbf{oz}(w\mathsf{0}) + 1$ is even, and $\mathsf{0}$ is a suffix of $w\mathsf{0}$.

- $(\mathsf{E}, \mathsf{1} \to \mathsf{E})$ Suppose $w \in \Lambda_{\mathsf{E}}$ and assume the inductive hypothesis, $\mathbf{zo}\, w$ is even, $\mathbf{oz}\, w$ is odd and $\mathsf{1}$ is a suffix of w. We have that $\mathbf{zo}(w\mathsf{1}) = \mathbf{zo}\, w$ is even, $\mathbf{oz}(w\mathsf{1}) = \mathbf{oz}\, w$ is odd, and $\mathsf{1}$ is a suffix of $w\mathsf{1}$.

- $(\mathsf{F}, \mathsf{0} \to \mathsf{G})$ Suppose $w \in \Lambda_{\mathsf{F}}$ and assume the inductive hypothesis, $\mathbf{zo}\, w$ is even, $\mathbf{oz}\, w$ is even and $\mathsf{1}$ is a suffix of w. We have that $\mathbf{zo}(w\mathsf{0}) = \mathbf{zo}\, w$ is even, $\mathbf{oz}(w\mathsf{0}) = \mathbf{oz}\, w + 1$ is odd, and $\mathsf{0}$ is a suffix of $w\mathsf{0}$.

- $(\mathsf{F}, 1 \rightarrow \mathsf{F})$ Suppose $w \in \Lambda_\mathsf{F}$ and assume the inductive hypothesis, $\mathbf{zo}\,w$ is even, $\mathbf{oz}\,w$ is even and 1 is a suffix of w. We have that $\mathbf{zo}(w1) = \mathbf{zo}\,w$ is even, $\mathbf{oz}(w1) = \mathbf{oz}\,w$ is even, and 1 is a suffix of $w1$.

- $(\mathsf{G}, 0 \rightarrow \mathsf{G})$ Suppose $w \in \Lambda_\mathsf{G}$ and assume the inductive hypothesis, $\mathbf{zo}\,w$ is even, $\mathbf{oz}\,w$ is odd and 0 is a suffix of w. We have that $\mathbf{zo}(w0) = \mathbf{zo}\,w$ is even, $\mathbf{oz}(w0) = \mathbf{oz}\,w$ is odd, and 0 is a suffix of $w0$.

- $(\mathsf{G}, 1 \rightarrow \mathsf{H})$ Suppose $w \in \Lambda_\mathsf{G}$ and assume the inductive hypothesis, $\mathbf{zo}\,w$ is even, $\mathbf{oz}\,w$ is odd and 0 is a suffix of w. We have that $\mathbf{zo}(w1) = \mathbf{zo}\,w + 1$ is odd, $\mathbf{oz}(w1) = \mathbf{oz}\,w$ is odd, and 1 is a suffix of $w1$.

- $(\mathsf{H}, 0 \rightarrow \mathsf{I})$ Suppose $w \in \Lambda_\mathsf{H}$ and assume the inductive hypothesis, $\mathbf{zo}\,w$ is odd, $\mathbf{oz}\,w$ is odd and 1 is a suffix of w. We have that $\mathbf{zo}(w0) = \mathbf{zo}\,w$ is odd, $\mathbf{oz}(w0) = \mathbf{oz}\,w + 1$ is even, and 0 is a suffix of $w0$.

- $(\mathsf{H}, 1 \rightarrow \mathsf{H})$ Suppose $w \in \Lambda_\mathsf{H}$ and assume the inductive hypothesis, $\mathbf{zo}\,w$ is odd, $\mathbf{oz}\,w$ is odd and 1 is a suffix of w. We have that $\mathbf{zo}(w1) = \mathbf{zo}\,w$ is odd, $\mathbf{oz}(w1) = \mathbf{oz}\,w$ is odd, and 1 is a suffix of $w1$.

- $(\mathsf{I}, 0 \rightarrow \mathsf{I})$ Suppose $w \in \Lambda_\mathsf{I}$ and assume the inductive hypothesis, $\mathbf{zo}\,w$ is odd, $\mathbf{oz}\,w$ is even and 0 is a suffix of w. We have that $\mathbf{zo}(w0) = \mathbf{zo}\,w$ is odd, $\mathbf{oz}(w0) = \mathbf{oz}\,w$ is even, and 0 is a suffix of $w0$.

- $(\mathsf{I}, 1 \rightarrow \mathsf{F})$ Suppose $w \in \Lambda_\mathsf{I}$ and assume the inductive hypothesis, $\mathbf{zo}\,w$ is odd, $\mathbf{oz}\,w$ is even and 0 is a suffix of w. We have that $\mathbf{zo}(w1) = \mathbf{zo}\,w + 1$ is even, $\mathbf{oz}(w1) = \mathbf{oz}\,w$ is even, and 1 is a suffix of $w1$.

□

Now, we use Lemma ES4.1.1 to prove that $L(M) = X$.

- $(L(M) \subseteq X)$ Suppose $w \in L(M)$. Hence $w \in L(M) = \bigcup\{\, \Lambda_q \mid q \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{E}, \mathsf{F}, \mathsf{G}, \mathsf{I}\}\,\}$, so that $w \in \Lambda_q$ for some $q \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{E}, \mathsf{F}, \mathsf{G}, \mathsf{I}\}$, and thus $w \in (\mathbf{alphabet}\,M)^* = \{0, 1\}^*$. Thus, there are seven cases to consider.
  - Suppose $q = \mathsf{A}$. Then $w \in \Lambda_\mathsf{A}$, so that $w = \%$, by Lemma ES4.1.1(A). Hence $\mathbf{zo}\,w = \mathbf{zo}\,\% = 0$ is even, so that $w \in X$.
  - Suppose $q = \mathsf{B}$. Then $w \in \Lambda_\mathsf{B}$, so that $\mathbf{zo}\,w$ is even, by Lemma ES4.1.1(B). Hence $w \in X$.
  - Suppose $q = \mathsf{C}$. Then $w \in \Lambda_\mathsf{C}$, so that $\mathbf{oz}\,w$ is even, by Lemma ES4.1.1(C). Hence $w \in X$.
  - Suppose $q = \mathsf{E}$. Then $w \in \Lambda_\mathsf{E}$, so that $\mathbf{zo}\,w$ is even, by Lemma ES4.1.1(E). Hence $w \in X$.
  - Suppose $q = \mathsf{F}$. Then $w \in \Lambda_\mathsf{F}$, so that $\mathbf{zo}\,w$ is even, by Lemma ES4.1.1(F). Hence $w \in X$.
  - Suppose $q = \mathsf{G}$. Then $w \in \Lambda_\mathsf{G}$, so that $\mathbf{zo}\,w$ is even, by Lemma ES4.1.1(G). Hence $w \in X$.
  - Suppose $q = \mathsf{I}$. Then $w \in \Lambda_\mathsf{I}$, so that $\mathbf{oz}\,w$ is even, by Lemma ES4.1.1(I). Hence $w \in X$.

- ($X \subseteq L(M)$) Suppose $w \in X$. Since $X \subseteq \{0,1\}^*$, we have that $w \in \{0,1\}^*$. Suppose, toward a contradiction, that $w \notin L(M)$. Thus $w \notin L(M) = \bigcup\{\,\Lambda_q \mid q \in \{\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{E},\mathsf{F},\mathsf{G},\mathsf{I}\}\,\}$. But $w \in \{0,1\}^* = (\textbf{alphabet}\,M)^* = \bigcup\{\,\Lambda_q \mid q \in Q_M\,\}$, so that $w \in \Lambda_\mathsf{D} \cup \Lambda_\mathsf{H}$. Thus there are two cases to consider.
  - Suppose $w \in \Lambda_\mathsf{D}$. Thus $\textbf{zo}\,w$ is odd and $\textbf{oz}\,w$ is odd, by Lemma ES4.1.1(D). Hence $w \notin X$—contradiction.
  - Suppose $w \in \Lambda_\mathsf{H}$. Thus $\textbf{zo}\,w$ is odd and $\textbf{oz}\,w$ is odd, by Lemma ES4.1.1(H). Hence $w \notin X$—contradiction.

  Because we achieved a contradiction in both cases, we have an overall contradiction. Thus $w \in L(M)$.

Exercise 2

(a) The regular expression α is $(\% + 0)12$. To see that our answer is correct, we put the description

```
{states}
A, B, C, D, E, F
{start state}
A
{accepting states}
F
{transitions}
A, % -> B | C;
B, % -> D;
C, 0 -> E;
E, % -> D;
D, 12 -> F
```

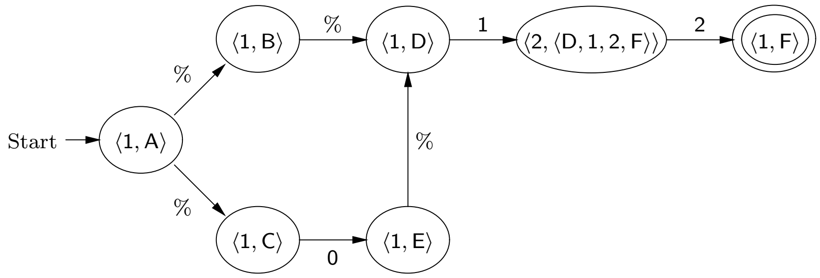
of the FA M_1 in the file `es4-ex2-fa`. We then load α and M_1 into Forlan, convert α to an FA, and check that it is isomorphic to M_1, as follows:

```
- val reg = Reg.input "";
@ (%+0)12
@ .
val reg = - : reg
- val fa1 = FA.input "es4-ex2-fa";
val fa1 = - : fa
- FA.isomorphic(regToFA reg, fa1);
val it = true : bool
```

(b) M_2 is formed from M_1 by the following process. First, the state A of M_1 is turned into the state $\langle 1,\mathsf{A}\rangle$ of M_2. $\langle 1,\mathsf{A}\rangle$ is the start state of M_2, since A is the start state of M_1; it is not an accepting state, since A is not an accepting state. Similarly, the non-accepting states B, C, D and E of M_1 are turned into the non-accepting states $\langle 1,\mathsf{B}\rangle$, $\langle 1,\mathsf{C}\rangle$, $\langle 1,\mathsf{D}\rangle$ and $\langle 1,\mathsf{E}\rangle$ of M_2. Next, the accepting state F of M_1 is turned into the accepting state $\langle 1,\mathsf{F}\rangle$ of M_2. Next, the transitions $(\mathsf{A},\%,\mathsf{B})$, $(\mathsf{A},\%,\mathsf{C})$, $(\mathsf{B},\%,\mathsf{D})$, $(\mathsf{C},0,\mathsf{E})$ and $(\mathsf{E},\%,\mathsf{D})$ of M_1 are turned into the transitions $(\langle 1,\mathsf{A}\rangle,\%,\langle 1,\mathsf{B}\rangle)$, $(\langle 1,\mathsf{A}\rangle,\%,\langle 1,\mathsf{C}\rangle)$,

$(\langle 1, \mathsf{B}\rangle, \%, \langle 1, \mathsf{D}\rangle)$, $(\langle 1, \mathsf{C}\rangle, 0, \langle 1, \mathsf{E}\rangle)$ and $(\langle 1, \mathsf{E}\rangle, \%, \langle 1, \mathsf{D}\rangle)$ of M_2. Finally, the transition $(\mathsf{D}, 12, \mathsf{F})$ of M_1 is split into the M_2 transitions $(\langle 1, \mathsf{D}\rangle, 1, \langle 2, \langle \mathsf{D}, 1, 2, \mathsf{F}\rangle\rangle)$ and $(\langle 2, \langle \mathsf{D}, 1, 2, \mathsf{F}\rangle\rangle, 2, \langle 1, \mathsf{F}\rangle)$, where $\langle 2, \langle \mathsf{D}, 1, 2, \mathsf{F}\rangle\rangle$ is a new, non-accepting state of M_2.

Here is a drawing of M_2:



Continuing our Forlan session, We convert the FA M_1 into an EFA, as follows:

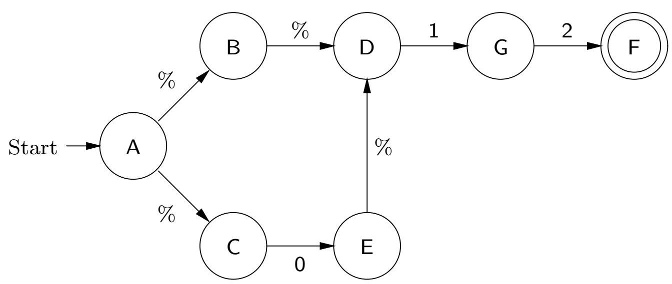
```
- val efa2 = faToEFA fa1;
val efa2 = - : efa
- EFA.output("", efa2);
{states} <1,A>, <1,B>, <1,C>, <1,D>, <1,E>, <1,F>, <2,<D,1,2,F>>
{start state} <1,A> {accepting states} <1,F>
{transitions}
<1,A>, % -> <1,B> | <1,C>; <1,B>, % -> <1,D>; <1,C>, 0 -> <1,E>;
<1,D>, 1 -> <2,<D,1,2,F>>; <1,E>, % -> <1,D>; <2,<D,1,2,F>>, 2 -> <1,F>
val it = () : unit
```

It is easy to check that M_2 is the outputted EFA.

(c) Continuing with our Forlan session, we rename the states of M_2, producing an EFA M_3, as follows:

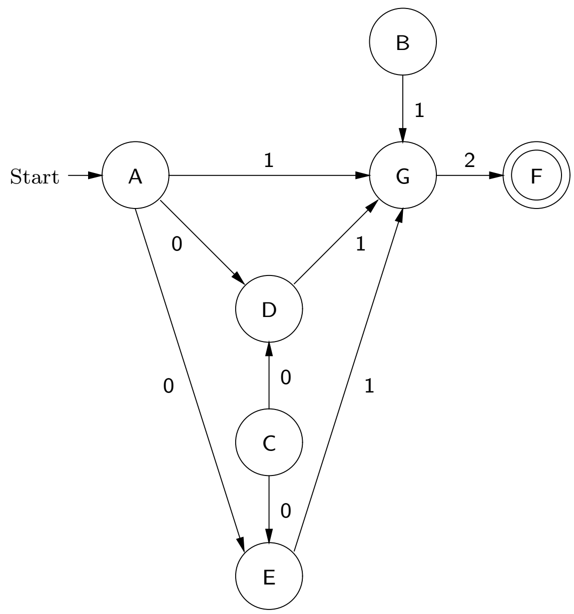
```
- val efa3 = EFA.renameStatesCanonically efa2;
val efa3 = - : efa
- EFA.output("", efa3);
{states} A, B, C, D, E, F, G {start state} A {accepting states} F
{transitions}
A, % -> B | C; B, % -> D; C, 0 -> E; D, 1 -> G; E, % -> D; G, 2 -> F
val it = () : unit
```

Here is a drawing of M_3:



(d) The NFA M_4 is formed from M_3 by the following process. Its states and start state are the same as those of M_3. The accepting states of M_4 are the elements of **emptyCloseBackwards** A_{M_3} = **emptyCloseBackwards** $\{\mathsf{F}\} = \{\mathsf{F}\}$. The transitions of M_4 are formed by processing the non-%-transitions of M_3. The transition $(\mathsf{C}, 0, \mathsf{E})$ of M_3 is turned into the transitions $(\mathsf{A}, 0, \mathsf{D})$, $(\mathsf{A}, 0, \mathsf{E})$, $(\mathsf{C}, 0, \mathsf{D})$ and $(\mathsf{C}, 0, \mathsf{E})$ of M_4, since **emptyCloseBackwards** $\{\mathsf{C}\} = \{\mathsf{A}, \mathsf{C}\}$ and **emptyClose** $\{\mathsf{E}\} = \{\mathsf{D}, \mathsf{E}\}$. The transition $(\mathsf{D}, 1, \mathsf{G})$ of M_3 is turned into the transitions $(\mathsf{A}, 1, \mathsf{G})$, $(\mathsf{B}, 1, \mathsf{G})$, $(\mathsf{D}, 1, \mathsf{G})$ and $(\mathsf{E}, 1, \mathsf{G})$ of M_4, since **emptyCloseBackwards** $\{\mathsf{D}\} = \{\mathsf{A}, \mathsf{B}, \mathsf{D}, \mathsf{E}\}$ and **emptyClose** $\{\mathsf{G}\} = \{\mathsf{G}\}$. Finally, the transition $(\mathsf{G}, 2, \mathsf{F})$ of M_3 is turned into the transition $(\mathsf{G}, 2, \mathsf{F})$ of M_4, since **emptyCloseBackwards** $\{\mathsf{G}\} = \{\mathsf{G}\}$ and **emptyClose** $\{\mathsf{F}\} = \{\mathsf{F}\}$.

Here is a drawing of M_4:



Continuing our Forlan session, we convert the EFA M_3 into an NFA, and output the result, as follows.

```
- val nfa4 = efaToNFA efa3;
val nfa4 = - : nfa
- NFA.output("", nfa4);
{states} A, B, C, D, E, F, G {start state} A {accepting states} F
{transitions}
A, 0 -> D | E; A, 1 -> G; B, 1 -> G; C, 0 -> D | E; D, 1 -> G; E, 1 -> G;
G, 2 -> F
val it = () : unit
```

It is easy to check that M_4 is the outputted NFA.

(e) We form the DFA M_5 from the NFA M_4, by the following process, which involves the construction of a set X of sets of states of M_4. Because the alphabet of M_4 is $\{0, 1, 2\}$, this will also be the alphabet of M_5. First, $\{\mathsf{A}\}$ is added to X and $\langle\mathsf{A}\rangle$ is made the start state of M_5, since A is M_4's start state. Since A is not an accepting state of M_4, $\langle\mathsf{A}\rangle$ is not an accepting state of M_5.

Since $\{\mathsf{A}\} \in X$ and $\Delta(\{\mathsf{A}\}, \mathsf{0}) = \{\mathsf{D}, \mathsf{E}\}$, we add $\{\mathsf{D}, \mathsf{E}\}$ to X, $\langle\mathsf{D}, \mathsf{E}\rangle$ to Q_{M_5} and $(\langle\mathsf{A}\rangle, \mathsf{0}, \langle\mathsf{D}, \mathsf{E}\rangle)$ to T_{M_5}. Since $\{\mathsf{A}\} \in X$ and $\Delta(\{\mathsf{A}\}, \mathsf{1}) = \{\mathsf{G}\}$, we add $\{\mathsf{G}\}$ to X, $\langle\mathsf{G}\rangle$ to Q_{M_5} and $(\langle\mathsf{A}\rangle, \mathsf{1}, \langle\mathsf{G}\rangle)$ to T_{M_5}. Since $\{\mathsf{A}\} \in X$ and $\Delta(\{\mathsf{A}\}, \mathsf{2}) = \emptyset$, we add $\emptyset$ to X, $\langle\rangle$ to Q_{M_5} and $(\langle\mathsf{A}\rangle, \mathsf{2}, \langle\rangle)$ to T_{M_5}.

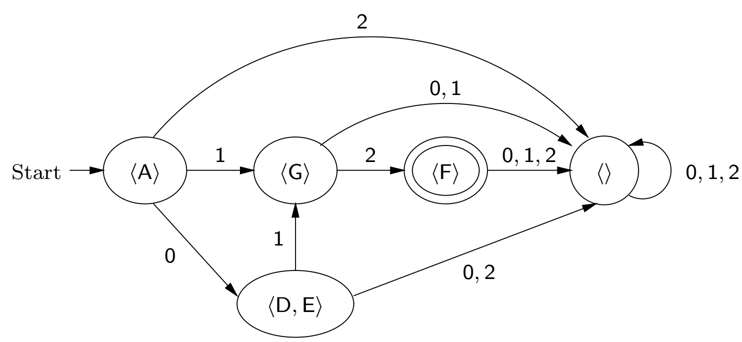
Since $\{\mathsf{D}, \mathsf{E}\} \in X$ and $\Delta(\{\mathsf{D}, \mathsf{E}\}, \mathsf{0}) = \emptyset \cup \emptyset = \emptyset$, and $\emptyset$ is already in X, we add $(\langle\mathsf{D}, \mathsf{E}\rangle, \mathsf{0}, \langle\rangle)$ to T_{M_5}. Since $\{\mathsf{D}, \mathsf{E}\} \in X$ and $\Delta(\{\mathsf{D}, \mathsf{E}\}, \mathsf{1}) = \{\mathsf{G}\} \cup \{\mathsf{G}\} = \{\mathsf{G}\}$, and $\{\mathsf{G}\}$ is already in X, we add $(\langle\mathsf{D}, \mathsf{E}\rangle, \mathsf{1}, \langle\mathsf{G}\rangle)$ to T_{M_5}. Since $\{\mathsf{D}, \mathsf{E}\} \in X$ and $\Delta(\{\mathsf{D}, \mathsf{E}\}, \mathsf{2}) = \emptyset \cup \emptyset = \emptyset$, and $\emptyset$ is already in X, we add $(\langle\mathsf{D}, \mathsf{E}\rangle, \mathsf{2}, \langle\rangle)$ to T_{M_5}.

Since $\{\mathsf{G}\} \in X$ and $\Delta(\{\mathsf{G}\}, \mathsf{0}) = \emptyset$, and $\emptyset$ is already in X, we add $(\langle\mathsf{G}\rangle, \mathsf{0}, \langle\rangle)$ to T_{M_5}. Since $\{\mathsf{G}\} \in X$ and $\Delta(\{\mathsf{G}\}, \mathsf{1}) = \emptyset$, and $\emptyset$ is already in X, we add $(\langle\mathsf{G}\rangle, \mathsf{1}, \langle\rangle)$ to T_{M_5}. Since $\{\mathsf{G}\} \in X$ and $\Delta(\{\mathsf{G}\}, \mathsf{2}) = \{\mathsf{F}\}$, we add $\{\mathsf{F}\}$ to X, $\langle\mathsf{F}\rangle$ to Q_{M_5} and $(\langle\mathsf{G}\rangle, \mathsf{2}, \langle\mathsf{F}\rangle)$ to T_{M_5}. Since F is an accepting state of M_4, $\langle\mathsf{F}\rangle$ is an accepting state of M_5.

Since $\emptyset \in X$ and $\Delta(\emptyset, \mathsf{0}) = \emptyset$, and $\emptyset$ is already in X, we add $(\langle\rangle, \mathsf{0}, \langle\rangle)$ to T_{M_5}. Since $\emptyset \in X$ and $\Delta(\emptyset, \mathsf{1}) = \emptyset$, and $\emptyset$ is already in X, we add $(\langle\rangle, \mathsf{1}, \langle\rangle)$ to T_{M_5}. Since $\emptyset \in X$ and $\Delta(\emptyset, \mathsf{2}) = \emptyset$, and $\emptyset$ is already in X, we add $(\langle\rangle, \mathsf{2}, \langle\rangle)$ to T_{M_5}.

Since $\{\mathsf{F}\} \in X$ and $\Delta(\{\mathsf{F}\}, \mathsf{0}) = \emptyset$, and $\emptyset$ is already in X, we add $(\langle\mathsf{F}\rangle, \mathsf{0}, \langle\rangle)$ to T_{M_5}. Since $\{\mathsf{F}\} \in X$ and $\Delta(\{\mathsf{F}\}, \mathsf{1}) = \emptyset$, and $\emptyset$ is already in X, we add $(\langle\mathsf{F}\rangle, \mathsf{1}, \langle\rangle)$ to T_{M_5}. Since $\{\mathsf{F}\} \in X$ and $\Delta(\{\mathsf{F}\}, \mathsf{2}) = \emptyset$, and $\emptyset$ is already in X, we add $(\langle\mathsf{F}\rangle, \mathsf{2}, \langle\rangle)$ to T_{M_5}.

Here is a drawing of M_5:



Continuing our Forlan session, we convert our NFA M_4 into a DFA, as follows:

```
- val dfa5 = nfaToDFA nfa4;
val dfa5 = - : dfa
- DFA.output("", dfa5);
{states} <>, <A>, <F>, <G>, <D,E> {start state} <A> {accepting states} <F>
{transitions}
<>, 0 -> <>; <>, 1 -> <>; <>, 2 -> <>; <A>, 0 -> <D,E>; <A>, 1 -> <G>;
<A>, 2 -> <>; <F>, 0 -> <>; <F>, 1 -> <>; <F>, 2 -> <>; <G>, 0 -> <>;
<G>, 1 -> <>; <G>, 2 -> <F>; <D,E>, 0 -> <>; <D,E>, 1 -> <G>; <D,E>, 2 -> <>
val it = () : unit
```

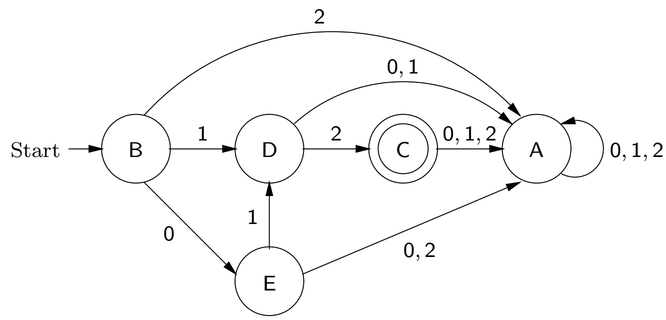
It is easy to check that M_5 is the outputted DFA.

(f) Continuing our Forlan session, we rename the states of M_5, producing a DFA M_6, as follows:

```
- val dfa6 = DFA.renameStatesCanonically dfa5;
```

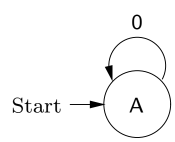
```
val dfa6 = - : dfa
- DFA.output("", dfa6);
{states} A, B, C, D, E {start state} B {accepting states} C
{transitions}
A, 0 -> A; A, 1 -> A; A, 2 -> A; B, 0 -> E; B, 1 -> D; B, 2 -> A; C, 0 -> A;
C, 1 -> A; C, 2 -> A; D, 0 -> A; D, 1 -> A; D, 2 -> C; E, 0 -> A; E, 1 -> D;
E, 2 -> A
val it = () : unit
```

Here is a drawing of M_6:



Exercise 3

We disprove the statement. Suppose, toward a contradiction, that the statement is true. Let $\mathit{simp} \in \mathbf{Reg} \to \mathbf{Reg}$ be the identity function and M be the FA

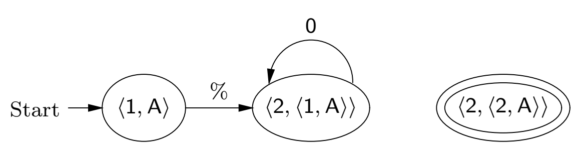


To see that, for all $\alpha \in \mathbf{Reg}$, $L(\mathit{simp}\,\alpha) = L(\alpha)$ and $\mathbf{alphabet}(\mathit{simp}\,\alpha) \subseteq \mathbf{alphabet}\,\alpha$, suppose $\alpha \in \mathbf{Reg}$. Then $L(\mathit{simp}\,\alpha) = L(\alpha)$ and $\mathbf{alphabet}(\mathit{simp}\,\alpha) = \mathbf{alphabet}\,\alpha \subseteq \mathbf{alphabet}\,\alpha$.

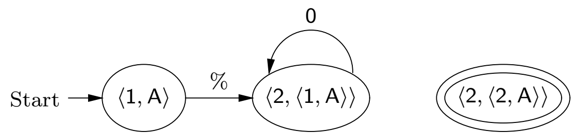
Thus, by the statement, we have that

$$\mathbf{alphabet}(\mathbf{faToReg}\,\mathit{simp}\,M) = \mathbf{alphabet}\,M = \{0\}.$$

According to the definition of **faToReg**, $\mathbf{faToReg}\,\mathit{simp}\,M$ is produced as follows. First, M is turned into the standard FA $M_1 = \mathbf{concat}(\mathbf{emptyStr}, \mathbf{concat}(M, \mathbf{emptyStr}))$:



Then, M_1 is turned into the standard RFA $M_2 = \mathbf{faToRFA}\,\mathit{simp}\,M_1$:



Next, the state $\langle 2, \langle 1, \mathsf{A}\rangle\rangle$ is removed from M_2, giving us the standard RFA M_3 = **removeState** *simp* $(M_2, \langle 2, \langle 1, \mathsf{A}\rangle\rangle)$:

(There are no transitions in M_3, because all of M_2's transitions involved $\langle 2, \langle 1, \mathsf{A}\rangle\rangle$, but there were no transitions from this state to another state.) Because M_3 has only its start and accepting states, and there is no transition from its start state to its accepting state, this means that **faToReg** *simp* M is \$. But then

$$\emptyset = \textbf{alphabet}\,\$ = \textbf{alphabet}(\textbf{faToReg}\, \mathit{simp}\, M) = \{\mathsf{0}\},$$

which is a contradiction. Thus the statement is false.