CIS 570 — Introduction to Formal Language Theory — Fall 2008

Exercise Set 1

Due by 4:00 p.m. on Tuesday, September 30

Exercise 1 (15 points)

Solve Exercise 1.1.2, proving that union distributes over intersection, i.e., for all sets A, B and C:

(1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. [8 points]

(2) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. [7 points]

Exercise 2 (15 points)

Solve Exercise 1.1.3, explaining what the three things are that are wrong with the following "definition".

> Let $f \in \mathbb{N} \to \mathbb{N}$ be the unique function such that, for all $n \in \mathbb{N}$,
>
> $$f\,n = \begin{cases} n - 2, & \text{if } n \geq 1 \text{ and } n \leq 10, \\ n + 2, & \text{if } n \geq 10. \end{cases}$$

Exercise 3 (15 points)

Proposition 1.2.2 shows that, for all $n \in \mathbb{N}$, $3n^2+3n+6$ is divisible by 6. Use this proposition to solve Exercise 1.2.3, using mathematical induction to prove that, for all $n \in \mathbb{N}$, $n(n^2+5)$ is divisible by 6.

Exercise 4 (15 points)

Solve Exercise 1.2.7, using strong induction to prove that, for all $n \in \mathbb{N}$, if $n \geq 1$, then there are $i, j \in \mathbb{N}$ such that $n = 2^i(2j+1)$.

Exercise 5 (40 points)

Define a function $\mathbf{diff} \in \{\mathsf{0},\mathsf{1}\}^* \to \mathbb{Z}$ by: for all $w \in \{\mathsf{0},\mathsf{1}\}^*$,

$$\mathbf{diff}\, w = \text{the number of } \mathsf{1}\text{'s in } w - 2(\text{the number of } \mathsf{0}\text{'s in } w).$$

Thus

- $\mathbf{diff}\,\% = 0$;
- $\mathbf{diff}\,\mathsf{0} = -2$;
- $\mathbf{diff}\,\mathsf{1} = 1$;
- for all $x, y \in \{\mathsf{0},\mathsf{1}\}^*$, $\mathbf{diff}(xy) = \mathbf{diff}\,x + \mathbf{diff}\,y$.

And, for all $w \in \{\mathsf{0},\mathsf{1}\}^*$, $\mathbf{diff}\,w = 0$ iff w has twice as many $\mathsf{1}$'s as $\mathsf{0}$'s.

Let X be the least subset of $\{\mathsf{0},\mathsf{1}\}^*$ such that:

(1) $\% \in X$;

(2) $\mathsf{1} \in X$;

(3) for all $x, y \in X$, $\mathsf{1}x\mathsf{1}y\mathsf{0} \in X$;

(4) for all $x, y \in X$, $xy \in X$.

Let $Y = \{\, w \in \{\mathsf{0},\mathsf{1}\}^* \mid \text{for all prefixes } v \text{ of } w, \mathbf{diff}\,v \geq 0 \,\}$.

(a) Prove that $X \subseteq Y$. Hint: use induction on X. [15 points]

(b) Prove that $Y \subseteq X$, completing the proof that $X = Y$. Hint: use strong string induction. [25 points]