Exercise Set 1
Due by 4:00 p.m. on Tuesday, September 30

Exercise 1 (15 points)
Solve Exercise 1.1.2, proving that union distributes over intersection, i.e., for all sets $A$, $B$ and $C$:

1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. [8 points]
2. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. [7 points]

Exercise 2 (15 points)
Solve Exercise 1.1.3, explaining what the three things are that are wrong with the following “definition”.

Let $f \in \mathbb{N} \to \mathbb{N}$ be the unique function such that, for all $n \in \mathbb{N}$,

$$f_n = \begin{cases} n - 2, & \text{if } n \geq 1 \text{ and } n \leq 10, \\ n + 2, & \text{if } n \geq 10. \end{cases}$$

Exercise 3 (15 points)
Proposition 1.2.2 shows that, for all $n \in \mathbb{N}$, $3n^2 + 3n + 6$ is divisible by 6. Use this proposition to solve Exercise 1.2.3, using mathematical induction to prove that, for all $n \in \mathbb{N}$, $n(n^2 + 5)$ is divisible by 6.

Exercise 4 (15 points)
Solve Exercise 1.2.7, using strong induction to prove that, for all $n \in \mathbb{N}$, if $n \geq 1$, then there are $i, j \in \mathbb{N}$ such that $n = 2^i(2j + 1)$.
Exercise 5 (40 points)

Define a function \( \text{diff} \in \{0, 1\}^* \rightarrow \mathbb{Z} \) by: for all \( w \in \{0, 1\}^* \),

\[
\text{diff} \ w = \text{the number of 1's in } w - 2(\text{the number of 0's in } w).
\]

Thus

- \( \text{diff} \ % = 0 \);
- \( \text{diff} \ 0 = -2 \);
- \( \text{diff} \ 1 = 1 \);
- for all \( x, y \in \{0, 1\}^* \), \( \text{diff} \ (xy) = \text{diff} \ x + \text{diff} \ y \).

And, for all \( w \in \{0, 1\}^* \), \( \text{diff} \ w = 0 \) iff \( w \) has twice as many 1's as 0's.

Let \( X \) be the least subset of \( \{0, 1\}^* \) such that:

1. \( \% \in X \);
2. \( 1 \in X \);
3. for all \( x, y \in X \), \( 1x1y0 \in X \);
4. for all \( x, y \in X \), \( xy \in X \).

Let \( Y = \{ w \in \{0, 1\}^* \mid \text{for all prefixes } v \text{ of } w, \text{diff} \ v \geq 0 \} \).

(a) Prove that \( X \subseteq Y \). Hint: use induction on \( X \). [15 points]

(b) Prove that \( Y \subseteq X \), completing the proof that \( X = Y \). Hint: use strong string induction. [25 points]