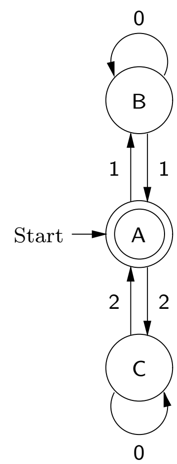
CIS 570 — Introduction to Formal Language Theory — Fall 2008

Mid-term Examination

Model Answers

Question 1

(a)



(b)

$$\Lambda_{\mathsf{A}} = X \quad \text{and} \quad \Lambda_{\mathsf{B}} = X\{\mathsf{1}\}\{\mathsf{0}\}^* \quad \text{and} \quad \Lambda_{\mathsf{C}} = X\{\mathsf{2}\}\{\mathsf{0}\}^*.$$

Question 2

$$\% + \mathsf{0} + (\mathsf{0} + \mathsf{1})^*(\mathsf{1} + \mathsf{10}) \quad \text{or} \quad (\mathsf{0}^*\mathsf{1})^*(\% + \mathsf{0})$$

Question 3

Since $L(\alpha) = \{\mathsf{0}, \mathsf{01}\}^*$, it will suffice to show that $\{\mathsf{0}, \mathsf{01}\}^* = X$. We will show that $\{\mathsf{0}, \mathsf{01}\}^* \subseteq X$ and $X \subseteq \{\mathsf{0}, \mathsf{01}\}^*$.

($\{\mathsf{0}, \mathsf{01}\}^* \subseteq X$) First we show that, for all $n \in \mathbb{N}$,

$$\{\mathsf{0}, \mathsf{01}\}^n \subseteq X.$$

We proceed by mathematical induction.

- (Basis Step) Since $\% \in \{\mathsf{0}, \mathsf{1}\}^*$, $\mathsf{1}$ is not a prefix of $\%$ and $\mathsf{011}$ is not a substring of $\%$, we have that $\% \in X$. Thus $\{\mathsf{0}, \mathsf{01}\}^0 = \{\%\} \subseteq X$.

- (Inductive Step) Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: $\{\mathsf{0}, \mathsf{01}\}^n \subseteq X$. We must show that $\{\mathsf{0}, \mathsf{01}\}^{n+1} \subseteq X$. By the inductive hypothesis, we have that

  $$\{\mathsf{0}, \mathsf{01}\}^{n+1} = \{\mathsf{0}, \mathsf{01}\}\{\mathsf{0}, \mathsf{01}\}^n \subseteq \{\mathsf{0}, \mathsf{01}\}X.$$

  Hence, it will suffice to show that $\{\mathsf{0}, \mathsf{01}\}X \subseteq X$. Suppose $w \in \{\mathsf{0}, \mathsf{01}\}X$. We must show that $w \in X$. Because $w \in \{\mathsf{0}, \mathsf{01}\}X$, we have that $w = xy$ for some $x \in \{\mathsf{0}, \mathsf{01}\}$ and $y \in X$. Since $x \in \{\mathsf{0}, \mathsf{01}\} \subseteq \{\mathsf{0}, \mathsf{1}\}^*$ and $y \in X \subseteq \{\mathsf{0}, \mathsf{1}\}^*$, we have that $w = xy \in \{\mathsf{0}, \mathsf{1}\}^*$. Because $x \in \{\mathsf{0}, \mathsf{01}\}$, there are two cases to consider.

  - Suppose $x = \mathsf{0}$. Then $w = \mathsf{0}y$, so that $\mathsf{1}$ is not a prefix of w. It remains to show that $\mathsf{011}$ is not a substring of $w = \mathsf{0}y$, i.e., that **(a)** $\mathsf{011}$ is not a prefix of $\mathsf{0}y$, and **(b)** $\mathsf{011}$ is not a substring of y. Part (a) holds, since $y \in X$, and thus $\mathsf{1}$ is not a prefix of y. And part (b) holds, since $y \in X$.
  - Suppose $x = \mathsf{01}$. Then $w = \mathsf{01}y$, so that $\mathsf{1}$ is not a prefix of w. It remains to show that $\mathsf{011}$ is not a substring of $w = \mathsf{01}y$, i.e., that **(a)** $\mathsf{011}$ is not a prefix of $\mathsf{01}y$, and **(b)** $\mathsf{011}$ is not a prefix of $\mathsf{1}y$, and **(c)** $\mathsf{011}$ is not a substring of y. Part (a) holds, since $y \in X$, and thus $\mathsf{1}$ is not a prefix of y. Part (b) obviously holds. And part (c) holds, since $y \in X$.

To see that $\{\mathsf{0}, \mathsf{01}\}^* \subseteq X$, suppose $w \in \{\mathsf{0}, \mathsf{01}\}^*$. Thus $w \in \{\mathsf{0}, \mathsf{01}\}^n$ for some $n \in \mathbb{N}$. Thus $w \in \{\mathsf{0}, \mathsf{01}\}^n \subseteq X$, by the result of our mathematical induction.

($X \subseteq \{\mathsf{0}, \mathsf{01}\}^*$) Because $X \subseteq \{\mathsf{0}, \mathsf{1}\}^*$, it will suffice to show that, for all $w \in \{\mathsf{0}, \mathsf{1}\}^*$,

$$\text{if } w \in X, \text{ then } w \in \{\mathsf{0}, \mathsf{01}\}^*.$$

We proceed by strong string induction. Suppose $w \in \{\mathsf{0}, \mathsf{1}\}^*$, and assume the inductive hypothesis: for all $x \in \{\mathsf{0}, \mathsf{1}\}^*$, if x is a proper substring of w, then

$$\text{if } x \in X, \text{ then } x \in \{\mathsf{0}, \mathsf{01}\}^*.$$

We must show that

$$\text{if } w \in X, \text{ then } w \in \{\mathsf{0}, \mathsf{01}\}^*.$$

Suppose $w \in X$. We must show that $w \in \{\mathsf{0}, \mathsf{01}\}^*$. There are three cases to consider.

- Suppose $w = \%$. Then $w = \% \in \{\mathsf{0}, \mathsf{01}\}^*$.
- Suppose $w = \mathsf{0}x$ for some $x \in \{\mathsf{0}, \mathsf{1}\}^*$. There are two subcases to consider.
  - Suppose $x \in X$. Since x is a proper substring of w, the inductive hypothesis tells us that $x \in \{\mathsf{0}, \mathsf{01}\}^*$. Thus $w = \mathsf{0}x \in \{\mathsf{0}, \mathsf{01}\}\{\mathsf{0}, \mathsf{01}\}^* \subseteq \{\mathsf{0}, \mathsf{01}\}^*$.
  - Suppose $x \notin X$. Since $\mathsf{0}x = w \in X$, it follows that $\mathsf{011}$ is not a substring of x. But $x \notin X$, and thus $\mathsf{1}$ is a prefix of x. Hence $w = \mathsf{01}y$ for some $y \in \{\mathsf{0}, \mathsf{1}\}^*$. Since $\mathsf{01}y = w \in X$, we have that $\mathsf{011}$ is not a prefix of $\mathsf{01}y$, so that $\mathsf{1}$ is not a prefix of y. And, $\mathsf{01}y \in X$ also tells us that $\mathsf{011}$ is not a substring of y, completing the proof that $y \in X$. Since y is a proper substring of w, the inductive hypothesis tells us that $y \in \{\mathsf{0}, \mathsf{01}\}^*$. Thus $w = (\mathsf{01})y \in \{\mathsf{0}, \mathsf{01}\}\{\mathsf{0}, \mathsf{01}\}^* \subseteq \{\mathsf{0}, \mathsf{01}\}^*$.
- Suppose $w = \mathsf{1}x$ for some $x \in \{\mathsf{0}, \mathsf{1}\}^*$. Thus $\mathsf{1}$ is a prefix of w. But $w \in X$, and thus $\mathsf{1}$ is not a prefix of w—contradiction. Thus $w \in \{\mathsf{0}, \mathsf{01}\}^*$.