Exercise 1

(a)

\[ 1 + (\% + 2 + 12)(012)^*(\% + 0 + 01) \]

(b) Let

\[ Y = \{1\} \cup \{\%, 2, 12\} \{012\}^* \{\%, 0, 01\}. \]

Since \( L(\alpha) = Y \), it will suffice to show that \( Y = X \).

Let

\[ A = \{\%\} \cup \{w \in X \mid 2 \text{ is a suffix of } w\}, \]
\[ B = \{\%\} \cup \{w \in X \mid 0 \text{ is a prefix of } w \text{ and } 2 \text{ is a suffix of } w\}. \]

Next, we prove a lemma that will help us show that \( Y \subseteq X \).

Lemma ES3.1.1

(1) \( \{012\} B \subseteq B \).

(2) \( \{012\}^* \subseteq B \).

(3) \( \{\%, 2, 12\} B \subseteq A \).

(4) \( A \{\%, 0, 01\} \subseteq X \).

Proof.

(1) Suppose \( w \in \{012\} B \). We must show that \( w \in B \). Since \( w \in \{012\} B \), we have that \( w = 012x \), for some \( x \in B \). Since \( x \in B \), there are two cases to consider. If \( x = \% \), then \( w = 012 \), so that \( w \in B \). So, suppose \( x \in X \), \( 0 \) is a prefix of \( x \), and \( 2 \) is a suffix of \( x \). Since \( x \in X \) and \( 0 \) is a prefix of \( x \), we have that \( w = 012x \in X \). Clearly, \( 0 \) is a prefix of \( 012x = w \). And, since \( 2 \) is a suffix of \( x \), we have that \( 2 \) is a suffix of \( 012x = w \). Hence \( w \in B \).

(2) It will suffice to show that, for all \( n \in \mathbb{N} \), \( \{012\}^n \subseteq B \). We proceed by mathematical induction.

(Basis Step) We have that \( \{012\}^0 = \{\%\} \subseteq B \).

(Inductive Step) Suppose \( n \in \mathbb{N} \), and assume the inductive hypothesis: \( \{012\}^n \subseteq B \). Then

\[ \{012\}^{n+1} = \{012\} \{012\}^n \]
\[ \subseteq \{012\} B \quad \text{(inductive hypothesis)} \]
\[ \subseteq B \quad \text{(Part (1))}. \]
Furthermore, 

\( w \in \{\%2, 12\}B \). We must show that \( w \in A \). Since \( w \in \{\%2, 12\}B \), we have that 

\( w = xy \), for some \( x \in \{\%2, 12\} \) and \( y \in B \). Because \( x \in \{\%2, 12\} \), there are three cases to consider:

- Suppose \( x = \% \). Then \( w = xy = \% y = y \in B \subseteq A \).
- Suppose \( x = 2 \). Then \( w = xy = 2y \). Since \( y \in B \), there are two subcases to consider. If \( y = \% \), then \( w = 2y = 2 \in A \). Otherwise, \( y \in X \), 0 is a prefix of \( y \), and 2 is a suffix of \( y \). Thus \( w = 2y \in X \) and 2 is a suffix of \( 2y = w \). Hence \( w \in A \).
- Suppose \( x = 12 \). Then \( w = xy = 12y \). Since \( y \in B \), there are two subcases to consider. If \( y = \% \), then \( w = 12y = 12 \in A \). Otherwise, \( y \in X \), 0 is a prefix of \( y \), and 2 is a suffix of \( y \). Thus \( w = 12y \in X \) and 2 is a suffix of \( 12y = w \). Hence \( w \in A \).

(4) Suppose \( w \in A\{\%, 0, 01\} \). We must show that \( w \in X \). Since \( w \in A\{\%, 0, 01\} \), we have that 

\( w = xy \), for some \( x \in A \) and \( y \in \{\%, 0, 01\} \). Since \( y \in \{\%, 0, 01\} \), there are three cases to consider:

- Suppose \( y = \% \). Then \( w = xy = x\% = x \in A \subseteq X \).
- Suppose \( y = 0 \). Then \( w = xy = x0 \). Since \( x \in A \), there are two subcases to consider. If \( x = \% \), then \( w = x0 = 0 \in X \). Otherwise, \( x \in X \) and 2 is a suffix of \( x \), so that \( w = x0 \in X \).
- Suppose \( y = 01 \). Then \( w = xy = x01 \). Since \( x \in A \), there are two subcases to consider. If \( x = \% \), then \( w = x01 = 01 \in X \). Otherwise, \( x \in X \) and 2 is a suffix of \( x \), so that \( w = x01 \in X \).

Now we can use the preceding lemma to show that \( Y \subseteq X \). We have that

\[
\begin{align*}
\{\%, 2, 12\}012\{\%, 0, 01\} & \subseteq \{\%, 2, 12\}B\{\%, 0, 01\} & \text{(Lemma ES3.1.1(2))} \\
& \subseteq A\{\%, 0, 01\} & \text{(Lemma ES3.1.1(3))} \\
& \subseteq X & \text{(Lemma ES3.1.1(4))}
\end{align*}
\]

Furthermore, \( 1 \in X \), so that

\[
\begin{align*}
Y & = \{1\} \cup \{\%, 2, 12\}012\{\%, 0, 01\} \subseteq X \cup X \\
& = X.
\end{align*}
\]

Hence \( Y \subseteq X \).

Finally, we show that \( X \subseteq Y \). Since \( X \subseteq \{0, 1, 2\}^* \), it will suffice to show that, for all \( w \in \{0, 1, 2\}^* \),

\[
\begin{align*}
\text{if } w \in X, \text{ then } w \in Y.
\end{align*}
\]

We proceed by strong string induction. Suppose \( w \in \{0, 1, 2\}^* \), and assume the inductive hypothesis: for all \( x \in \{0, 1, 2\}^* \), if \( |x| < |w| \), then

\[
\begin{align*}
\text{if } x \in X, \text{ then } x \in Y.
\end{align*}
\]

"
We must show that

\[ w \in X, \text{ then } w \in Y. \]

Suppose \( w \in X \). We must show that \( w \in Y \). There are four cases to consider.

- Suppose \( w = \% \). Then \( w = \% \in \{\%, 2, 12\}^2 \{\% , 0, 01\} \subseteq Y \).

- Suppose \( w = 0x \), for some \( x \in \{0, 1, 2\}^* \). If \( x = \% \), then \( w = 0 = \% \in \{\%, 2, 12\}^2 \{\% , 0, 01\} \subseteq Y \). So, suppose \( x \neq \% \). Since \( 0x = w \in X \), it follows that \( x = 1y \), for some \( y \in \{0, 1, 2\}^* \). Thus \( w = 01y \). If \( y = \% \), then \( w = 01 = \% \{01\} \in \{\%, 2, 12\}^2 \{\% , 0, 01\} \subseteq Y \). So, suppose \( y \neq \% \). Since \( 01y = w \in X \), it follows that \( y = 2z \), for some \( z \in \{0, 1, 2\}^* \). Thus \( w = 012z \). Because \( 012z = w \in X \), we have that \( 12z \in X \). Furthermore, since \( |12z| < |w| \), the inductive hypothesis tells us that \( 12z \in Y \). Because \( 12z \neq 1 \), we have that \( 12z = twv \), for some \( t \in \{\%, 2, 12\}, u \in \{\% , 0, 01\} \) and \( v \in \{\%, 0, 01\} \).

Since \( t \in \{\%, 2, 12\} \), there are three subcases to consider.

  - Suppose \( t = \% \). Thus \( 12z = twv = \% uv = uv \). Because \( u \in \{\% , 01\} \), it follows that \( u = \% \).
    Hence \( 12z = uv = \% v = v \). But \( v \in \{\%, 0, 01\} \), and none of the elements of \( \{\%, 0, 01\} \) begin with \( 1 \)—contradiction. Thus \( w \in Y \).

  - Suppose \( t = 2 \). Since \( 12z = twv \), it follows that \( 1 = 2 \)—contradiction. Thus \( w \in Y \).

  - Suppose \( t = 12 \). Thus \( w = 012z = 0(12z) = twv = (012)uv = \% (012)uv \in \{\%, 2, 12\} \{\% , 0, 01\} \subseteq \{\%, 2, 12\}^2 \{\% , 0, 01\} \subseteq Y \).

- Suppose \( w = 1x \), for some \( x \in \{0, 1, 2\}^* \). If \( x = \% \), then \( w = 1x = 1 \in Y \). So, suppose \( x \neq \% \). Since \( 1x = w \in X \), it follows that \( x = 2y \), for some \( y \in \{0, 1, 2\} \). Thus \( w = 12y \). Because \( 12y = w \in X \), we have that \( 2y \in X \). Furthermore, since \( |2y| < |w| \), the inductive hypothesis tells us that \( 2y \in Y \). Because \( 2y \neq 1 \), we have that \( 2y = twv \), for some \( t \in \{\%, 2, 12\}, u \in \{\% , 0, 01\} \) and \( v \in \{\%, 0, 01\} \).

  - Suppose \( t = \% \). Thus \( 2y = twv = \% uv = uv \). Because \( u \in \{\% , 01\} \), it follows that \( u = \% \).
    Hence \( 2y = uv = \% v = v \). But \( v \in \{\%, 0, 01\} \), and none of the elements of \( \{\%, 0, 01\} \) begin with \( 2 \)—contradiction. Thus \( w \in Y \).

  - Suppose \( t = 2 \). Thus \( w = 12y = 1(2y) = 1twv = (12)uv \in \{\%, 2, 12\} \{\% , 0, 01\} \subseteq Y \).

  - Suppose \( t = 12 \). Since \( 2y = twv \), it follows that \( 2 = 1 \)—contradiction. Thus \( w \in Y \).

- Suppose \( w = 2x \), for some \( x \in \{0, 1, 2\}^* \). If \( x = \% \), then \( w = 2x = 2 = 2\% \in \{\%, 2, 12\} \{\% , 0, 01\} \subseteq Y \). So, suppose \( x \neq \% \). Since \( 2x = w \in X \), it follows that \( x = 0y \), for some \( y \in \{0, 1, 2\} \). Thus \( w = 20y \). Because \( 20y = w \in X \), we have that \( 0y \in X \). Furthermore, since \( 0y \neq 0 \), the inductive hypothesis tells us that \( 0y \in Y \). Because \( 0y \neq 1 \), we have that \( 0y = twv \), for some \( t \in \{\%, 2, 12\}, u \in \{\% , 0, 01\} \) and \( v \in \{\%, 0, 01\} \). Since \( 0y = twv \) and \( t \in \{\%, 2, 12\} \), it follows that \( t = \% \). Thus \( w = 20y = 2(0y) = 2twv = 2w \in \{\%, 2, 12\} \{\% , 0, 01\} \subseteq Y \).

Because \( Y \subseteq X \subseteq Y \), we have that \( Y = X \).
Exercise 2

(a)

(b) First, we put the following text, which describes $M$ in Forlan’s syntax, in the file `es3-ex2-fa`:

```plaintext
{states}
A, B, C
{start state}
A
{accepting states}
A, B
{transitions}
A, 0 -> B; A, 1 -> A;
B, 0 -> C; B, 1 -> A;
C, 0 -> C; C, 10 -> C; C, 11 -> A
```

Next, we load $M$ into Forlan, and call it `fa`:

```plaintext
val fa = FA.input "es3-ex2-fa";
val fa = - : fa
```

Next, we find minimum-length labeled paths explaining why some strings in $X$ are accepted by `fa` ($M$):

```plaintext
val findAcceptingLP = FA.findAcceptingLP fa;
val findAcceptingLP = fn : str -> lp
  fun test s =
    (print(s ^ "\n");
     LP.output("", findAcceptingLP(Str.fromString s)));
val test = fn : string -> unit
  app test
    ["%", "1", "0", "11", "101", "0101", "0111", "00110", "110", "01000100110", "0011010", "00011010", "00010100011"];
```

%:
A
I:
A, 1 => A
0:
A, 0 => B
I1:
A, 1 => A, 1 => A
Finally, we check that some strings that are not in $X$ are not accepted by $fa (M)$:

val accepted = FA.accepted fa;
val accepted = fn : str -> bool
val test = fn : string -> string * bool
val it = [("00",false),("001",false),("0010",false),("000101",false),"000",false),
          ("010001100",false),("1100",false),("1010001100",false),
          ("1000011010010",false)] : (string * bool) list

(e) First we prove some basic properties about $X$.

Lemma ES3.2.1
1. For all $w \in \{0,1\}^*$, if $00$ is not a substring of $w$, then $w \in X$.
2. $\{0,1\}^* \{11\} \subseteq X$.
3. $X \{1\} \subseteq X$.

Proof.
1. Suppose $w \in \{0,1\}^*$ and $00$ is not a substring of $w$. To see that $w \in X$, suppose $x, y \in \{0,1\}^*$ and $w = x00y$. We must show that $11$ is a substring of $y$. Then $00$ is a substring of $w$—contradiction. Thus $11$ is a substring of $y$. 
(2) Suppose \( w \in \{0,1\}^* \{11\} \). Hence \( w = x11 \) for some \( x \in \{0,1\}^* \). To see that \( w \in X \), suppose \( u, v \in \{0,1\}^* \) and \( w = u00v \). We must show that \( 11 \) is a substring of \( v \). We have that \( u00v = w = x11 \). Thus \( 11 \) is a suffix, and thus a substring, of \( v \).

(3) Suppose \( w \in X \{1\} \). Thus \( w = x1 \) for some \( x \in X \). To see that \( w \in X \), suppose \( u, v \in \{0,1\}^* \) and \( w = u00v \). We must show that \( 11 \) is a substring of \( v \). We have that \( u00v = w = x1 \). Thus \( v = t1 \) for some \( t \in \{0,1\}^* \). Because \( x1 = u00v = u00t1 \), we have that \( x = u00t \). Because \( x \in X \), it follows that \( 11 \) is a substring of \( t \). But \( v = t1 \), and thus \( 11 \) is a substring of \( v \).

\( q \)

By Lemma ES3.2.1(1), we have that \( %, 0 \in X \). Define languages \( A, B \) and \( C \) by:

\[
A = \{ w \in X \mid 0 \text{ is not a suffix of } w \},
B = \{ w \in X \mid 0 \text{ is a suffix of } w \},
C = \{ w \in \{0,1\}^* \mid w \not\in X \text{ and } 0 \text{ is a suffix of } w \}.
\]

It is easy to see that \( X = A \cup B \). Next, we prove a useful lemma that is derived from the definition of \( M \).

**Lemma ES3.2.2**

For all \( w \in \{0,1\}^* \):

(A) \( w \in A \) iff \( w = %, \) or \( w = x1 \) for some \( x \in A \), or \( w = x1 \) for some \( x \in B \), or \( w = x11 \) for some \( x \in C \);

(B) \( w \in B \) iff \( w = x0 \) for some \( x \in A \);

(C) \( w \in C \) iff \( w = x0 \) for some \( x \in B \), or \( w = x0 \) for some \( x \in C \), or \( w = x10 \) for some \( x \in C \).

**Proof.**

(A) ("only if" direction) Suppose \( w \in A \). Thus \( w \in X \) and \( 0 \) is not a suffix of \( w \). If \( w = % \), then we are done, so suppose \( w \neq % \). Because \( 0 \) is not a suffix of \( w \), we have that \( w = x1 \) for some \( x \in \{0,1\}^* \). There are two cases to consider.

- Suppose \( x \in X \). Because \( X = A \cup B \), we have that \( x \in A \) or \( x \in B \), and thus either \( w = x1 \) and \( x \in A \), or \( w = x1 \) and \( x \in B \).
- Suppose \( x \not\in X \). Thus \( x = u00v \) for some \( u, v \in \{0,1\}^* \) such that \( 11 \) is not a substring of \( v \). Hence \( w = x1 = u00v1 \). Because \( u00v1 = w \in X \), it follows that \( 11 \) is a substring of \( v1 \). But \( 11 \) is not a substring of \( v \), and thus \( 1 \) is a suffix of \( v \). Thus \( v = t1 \) for some \( t \in \{0,1\}^* \), so that \( x = u00v = u00t1 \). Hence \( w = x1 = u00/11 \). Because \( 11 \) is not a substring of \( v \), it follows that \( 11 \) is not a substring of \( t \) and \( 1 \) is not a suffix of \( t \). Thus \( u00t \not\in X \). And, either \( t = % \) or \( 0 \) is a suffix of \( t \). Hence we have that \( 0 \) is a suffix of \( u00t \). Thus \( w = (u00t)11 \) and \( u00t \in C \).

("if" direction) Suppose \( w = %, \) or \( w = x1 \) for some \( x \in A \), or \( w = x1 \) for some \( x \in B \), or \( w = x11 \) for some \( x \in C \). There are four cases to consider.
• Suppose \( w = \% \). Then \( w = \% \in X \). And 0 is not a substring of \( \% = w \), completing the proof that \( w \in A \).

• Suppose \( w = x1 \) for some \( x \in A \). Thus \( x \in X \). By Lemma ES3.2.1(3), we have that \( w = x1 \in X \{1\} \subseteq X \). And 0 is not a suffix of \( x1 = w \), completing the proof that \( w \notin A \).

• Suppose \( w = x1 \) for some \( x \in B \). Thus \( x \in X \). By Lemma ES3.2.1(3), we have that \( w = x1 \in X \{1\} \subseteq X \). And 0 is not a suffix of \( x1 = w \), completing the proof that \( w \in A \).

• Suppose \( w = x11 \) for some \( x \in C \). By Lemma ES3.2.1(2), we have that \( w = x11 \in \{0,1\}\{1\} \subseteq X \). And 0 is not a suffix of \( x11 = w \), completing the proof that \( w \in A \).

(B) ("only if" direction) Suppose \( w \in B \). Thus \( w \in X \) and 0 is a suffix of \( w \). Hence \( w = x0 \) for some \( x \in \{0,1\}^* \).

Suppose, toward a contradiction, that 0 is a suffix of \( x \). Then \( x = y0 \) for some \( y \in \{0,1\}^* \), so that \( w = x0 = y00\% \). But then \( w \notin X \) — contradiction. Thus 0 is not a suffix of \( x \).

Suppose, toward a contradiction, that \( x \notin X \). Thus \( x = u00v \) for some \( u, v \in \{0,1\}^* \) such that \( 11 \) is not a substring of \( v \). Because \( u00v0 = x0 = w \in X \), we have that \( 11 \) is a substring of \( v0 \), so that \( 11 \) is a substring of \( v \)—contradiction. Thus \( x \in X \), completing the proof that \( x \in A \).

Hence \( w = x0 \) and \( x \in A \).

("if" direction) Suppose \( w = x0 \) for some \( x \in A \). Thus \( x \in X \) and 0 is not a suffix of \( x \).

To see that \( w \in X \), suppose that \( u, v \in \{0,1\}^* \) and \( w = u00v \). We must show that \( 11 \) is a substring of \( v \). We have that \( x0 = w = u00v \). Because 0 is not a suffix of \( x \), we have that \( v \neq \% \). Thus \( v = t0 \) for some \( t \in \{0,1\}^* \). Hence \( x0 = u00v = u00t0 \), so that \( x = u00t \).

Because \( x \in X \), it follows that \( 11 \) is a substring of \( t \). But \( v = t0 \), and thus \( 11 \) is a substring of \( v \).

Finally, 0 is a suffix of \( x0 = w \), completing the proof that \( w \in B \).

(C) ("only if" direction) Suppose \( w \in C \). Thus \( w \in \{0,1\}^* \), \( w \notin X \) and 0 is a suffix of \( w \), so that \( w = x0 \) for some \( x \in \{0,1\}^* \). If \( x = \% \), then \( w = 0 \in X \) — contradiction. Thus \( x \neq \% \).

There are two cases to consider.

• Suppose \( x = y0 \) for some \( y \in \{0,1\}^* \). Thus 0 is a suffix of \( x \). There are two subcases to consider.

  - Suppose \( x \in X \). Thus \( w = x0 \) and \( x \in B \).

  - Suppose \( x \notin X \). Thus \( w = x0 \) and \( x \in C \).

• Suppose \( x = y1 \) for some \( y \in \{0,1\}^* \). Hence \( w = x0 = y10 \). Because \( w \notin X \), there are \( u, v \in \{0,1\}^* \) such that \( w = u00v \) and \( 11 \) is not a substring of \( v \). Thus \( y10 = w = u00v \). Clearly \( v \neq \% \), and so 0 is a suffix of \( v \). And \( v \) cannot be 0, and thus \( v = t10 \) for some \( t \in \{0,1\}^* \). Because \( y10 = u00v = u00t10 \), we have that \( y = u00t \). Since \( 11 \) is not a substring of \( v = t10 \), it follows that \( 11 \) is not a substring of \( t \), and 1 is not a suffix of \( t \).

Hence \( y = u00t \notin X \). Because 1 is not a suffix of \( t \), either \( t = \% \) or 0 is a suffix of \( t \). In either case, we have that \( 0 \) is suffix of \( u00t = y \), completing the proof that \( y \in C \). Thus \( w = y10 \) and \( y \in C \).
("if" direction) Suppose \( w = x0 \) for some \( x \in B \), or \( w = x0 \) for some \( x \in C \), or \( w = x10 \) for some \( x \in C \). There are three cases to consider.

- Suppose \( w = x0 \) for some \( x \in B \). Thus 0 is a suffix of \( x \), so that 00 is a suffix of \( x0 = w \). Hence \( w = u00\% \) for some \( u \in \{0, 1\}^* \), so that \( w \notin X \). And 0 is a suffix of \( x0 = w \), completing the proof that \( w \in C \).

- Suppose \( w = x0 \) for some \( x \in C \). Thus 0 is a suffix of \( x \), so that 00 is a suffix of \( x0 = w \). Hence \( w = u00\% \) for some \( u \in \{0, 1\}^* \), so that \( w \notin X \). And 0 is a suffix of \( x0 = w \), completing the proof that \( w \in C \).

- Suppose \( w = x10 \) for some \( x \in C \). Thus 0 is a suffix of \( x \) and \( x \notin X \), i.e., \( x = u00v \) for some \( u, v \in \{0, 1\}^* \) such that 11 is not a substring of \( v \). Because 0 is a suffix of \( x \), and \( x = u00v \), it follows that 1 is not a suffix of \( v \). Hence 11 is not a substring of \( v10 \). But \( w = x10 = u00v10 \), and thus \( w \notin X \). And 0 is a suffix of \( x10 = w \), completing the proof that \( w \in C \).

\[ \square \]

Now we prove a lemma that will allow us to establish that \( L(M) \subseteq X \).

**Lemma ES3.2.3**

For all \( w \in \{0, 1\}^* \):

(A) if \( A \in \Delta(\{A\}, w) \), then \( w \in A \);

(B) if \( B \in \Delta(\{A\}, w) \), then \( w \in B \);

(C) if \( C \in \Delta(\{A\}, w) \), then \( w \in C \).

**Proof.** We proceed by strong string induction. Suppose \( w \in \{0, 1\}^* \), and assume the inductive hypothesis: for all \( x \in \{0, 1\}^* \), if \( |x| < |w| \), then:

(A) if \( A \in \Delta(\{A\}, x) \), then \( x \in A \);

(B) if \( B \in \Delta(\{A\}, x) \), then \( x \in B \);

(C) if \( C \in \Delta(\{A\}, x) \), then \( x \in C \).

We must show that:

(A) if \( A \in \Delta(\{A\}, w) \), then \( w \in A \);

(B) if \( B \in \Delta(\{A\}, w) \), then \( w \in B \);

(C) if \( C \in \Delta(\{A\}, w) \), then \( w \in C \).

We proceed as follows.

(A) Suppose \( A \in \Delta(\{A\}, w) \). We must show that \( w \in A \). Since \( A \in \Delta(\{A\}, w) \), there are two cases to consider.

- Suppose \( A = A \) and \( w = \% \). By Lemma ES3.2.2(A), \( w \in A \).
Suppose there are $q \in Q$ and $x, y \in \text{Str}$ such that $w = xy$, $q \in \Delta([A], x)$ and $(q, y, A) \in T$. Since $(q, y, A) \in T$, there are three cases to consider.

- Suppose $q = A$ and $y = 1$. Thus $A \in \Delta([A], x)$ and $w = xy = x1$. Since $|x| < |w|$, Part (A) of the inductive hypothesis tells us that $x \in A$. Hence, by Lemma ES3.2.2(A), we have that $w \in A$.

- Suppose $q = B$ and $y = 1$. Thus $B \in \Delta([A], x)$ and $w = xy = x1$. Since $|x| < |w|$, Part (B) of the inductive hypothesis tells us that $x \in B$. Hence, by Lemma ES3.2.2(A), we have that $w \in A$.

- Suppose $q = C$ and $y = 11$. Thus $C \in \Delta([A], x)$ and $w = xy = x11$. Since $|x| < |w|$, Part (C) of the inductive hypothesis tells us that $x \in C$. Hence, by Lemma ES3.2.2(A), we have that $w \in A$.

(B) Suppose $B \in \Delta([A], w)$. We must show that $w \in B$. Since $B \in \Delta([A], w)$, there are $q \in Q$ and $x, y \in \text{Str}$ such that $w = xy$, $q \in \Delta([A], x)$ and $(q, y, B) \in T$. Since $(q, y, B) \in T$, we have that $q = A$ and $y = 0$. Thus $A \in \Delta([A], x)$ and $w = xy = x0$. Since $|x| < |w|$, Part (A) of the inductive hypothesis tells us that $x \in A$. Hence, by Lemma ES3.2.2(B), we have that $w \in B$.

(C) Suppose $C \in \Delta([A], w)$. We must show that $w \in C$. Since $C \in \Delta([A], w)$, there are $q \in Q$ and $x, y \in \text{Str}$ such that $w = xy$, $q \in \Delta([A], x)$ and $(q, y, C) \in T$. Since $(q, y, C) \in T$, there are three cases to consider.

- Suppose $q = B$ and $y = 0$. Thus $B \in \Delta([A], x)$ and $w = xy = x0$. Since $|x| < |w|$, Part (B) of the inductive hypothesis tells us that $x \in B$. Hence, by Lemma ES3.2.2(C), we have that $w \in C$.

- Suppose $q = C$ and $y = 0$. Thus $C \in \Delta([A], x)$ and $w = xy = x0$. Since $|x| < |w|$, Part (C) of the inductive hypothesis tells us that $x \in C$. Hence, by Lemma ES3.2.2(C), we have that $w \in C$.

- Suppose $q = C$ and $y = 10$. Thus $C \in \Delta([A], x)$ and $w = xy = x10$. Since $|x| < |w|$, Part (C) of the inductive hypothesis tells us that $x \in C$. Hence, by Lemma ES3.2.2(C), we have that $w \in C$.

Now, we use the preceding lemma to show that $I(M) \subseteq X$. Suppose $w \in I(M)$. We must show that $w \in X$. Because $w \in I(M)$, we have that $\Delta([A], w) \cap \{A, B\} \neq \emptyset$, so that $A \in \Delta([A], w)$ or $B \in \Delta([A], w)$. Since alphabet($M$) = $\{0, 1\}$, we have that $w \in \{0, 1\}^*$. Thus, we have that Parts (A)--(C) of Lemma ES3.2.3 hold. By Parts (A) and (B), it follows that $w \in A$ or $w \in B$. But $A \subseteq X$ and $B \subseteq X$, and thus $w \in X$.

Next, we prove a lemma that will allow us to establish that $X \subseteq I(M)$.

**Lemma ES3.2.4**

*For all $w \in \{0, 1\}^*$:

(A) if $w \in A$, then $A \in \Delta([A], w)$;*
(B) if \( w \in B \), then \( B \in \Delta(\{A\}, w) \);

(C) if \( w \in C \), then \( C \in \Delta(\{A\}, w) \).

**Proof.** We proceed by strong string induction. Suppose \( w \in \{0,1\}^* \), and assume the inductive hypothesis: for all \( x \in \{0,1\}^* \), if \( |x| < |w| \), then:

(A) if \( x \in A \), then \( A \in \Delta(\{A\}, x) \);

(B) if \( x \in B \), then \( B \in \Delta(\{A\}, x) \);

(C) if \( x \in C \), then \( C \in \Delta(\{A\}, x) \).

We must show that:

(A) if \( w \in A \), then \( A \in \Delta(\{A\}, w) \);

(B) if \( w \in B \), then \( B \in \Delta(\{A\}, w) \);

(C) if \( w \in C \), then \( C \in \Delta(\{A\}, w) \).

We proceed as follows.

(A) Suppose \( w \in A \). We must show that \( A \in \Delta(\{A\}, w) \). By Lemma ES3.2.2(A), there are four cases to consider.

- Suppose \( w = \% \). We have that \( A \in \Delta(\{A\}, \%) \), and thus that \( A \in \Delta(\{A\}, w) \).
- Suppose \( w = x1 \) for some \( x \in A \). Since \( |x| < |w| \), Part (A) of the inductive hypothesis tells us that \( A \in \Delta(\{A\}, x) \). Since \( (A, 1, A) \in T \), we have that \( A \in \Delta(\{A\}, 1) \). Hence \( A \in \Delta(\{A\}, x1) \), i.e., \( A \in \Delta(\{A\}, w) \).
- Suppose \( w = x1 \) for some \( x \in B \). Since \( |x| < |w| \), Part (B) of the inductive hypothesis tells us that \( B \in \Delta(\{A\}, x) \). Since \( (B, 1, A) \in T \), we have that \( A \in \Delta(\{B\}, 1) \). Hence \( A \in \Delta(\{A\}, x1) \), i.e., \( A \in \Delta(\{A\}, w) \).
- Suppose \( w = x11 \) for some \( x \in C \). Since \( |x| < |w| \), Part (C) of the inductive hypothesis tells us that \( C \in \Delta(\{A\}, x) \). Since \( (C, 11, A) \in T \), we have that \( A \in \Delta(\{C\}, 11) \). Hence \( A \in \Delta(\{A\}, x11) \), i.e., \( A \in \Delta(\{A\}, w) \).

(B) Suppose \( w \in B \). We must show that \( B \in \Delta(\{A\}, w) \). By Lemma ES3.2.2(B), we have that \( w = x0 \) for some \( x \in A \). Since \( |x| < |w| \), Part (A) of the inductive hypothesis tells us that \( A \in \Delta(\{A\}, x) \). Since \( (A, 0, B) \in T \), we have that \( B \in \Delta(\{A\}, 0) \). Hence \( B \in \Delta(\{A\}, x0) \), i.e., \( B \in \Delta(\{A\}, w) \).

(C) Suppose \( w \in C \). We must show that \( C \in \Delta(\{A\}, w) \). By Lemma ES3.2.2(C), there are three cases to consider.

- Suppose \( w = x0 \) for some \( x \in B \). Since \( |x| < |w| \), Part (B) of the inductive hypothesis tells us that \( B \in \Delta(\{A\}, x) \). Since \( (B, 0, C) \in T \), we have that \( C \in \Delta(\{B\}, 0) \). Hence \( C \in \Delta(\{A\}, x0) \), i.e., \( C \in \Delta(\{A\}, w) \).
Suppose \( w = x0 \) for some \( x \in C \). Since \(|x| < |w|\), Part (C) of the inductive hypothesis tells us that \( C \in \Delta(\{A\}, x) \). Since \((C, 0, C) \in T\), we have that \( C \in \Delta(\{C\}, 0) \). Hence \( C \in \Delta(\{A\}, x0) \), i.e., \( C \in \Delta(\{A\}, w) \).

Suppose \( w = x10 \) for some \( x \in C \). Since \(|x| < |w|\), Part (C) of the inductive hypothesis tells us that \( C \in \Delta(\{A\}, x) \). Since \((C, 10, C) \in T\), we have that \( C \in \Delta(\{C\}, 10) \). Hence \( C \in \Delta(\{A\}, x10) \), i.e., \( C \in \Delta(\{A\}, w) \).

\( \square \)

Now, we use the preceding lemma to show that \( X \subseteq L(M) \). Suppose \( w \in X \). We must show that \( w \in L(M) \). Since \( X \subseteq \{0, 1\}^* \), we have that Parts (A)–(C) of Lemma ES3.2.4 hold. There are two cases to consider.

1. Suppose \( 0 \) is a suffix of \( w \). Then \( w \in B \), so that \( B \in \Delta(\{A\}, w) \), by Part (B). Thus \( \Delta(\{A\}, w) \cap \{A, B\} \neq \emptyset \), showing that \( w \in L(M) \).

2. Suppose \( 0 \) is not a suffix of \( w \). Then \( w \in A \), so that \( A \in \Delta(\{A\}, w) \), by Part (A). Thus \( \Delta(\{A\}, w) \cap \{A, B\} \neq \emptyset \), showing that \( w \in L(M) \).

Because \( L(M) \subseteq X \subseteq L(M) \), we have that \( L(M) = X \). Finally, we must show that \( M \) has as few states as possible.

**Lemma ES3.2.5**

For all FAs \( N \), if \( L(N) = X \) and \( |Q_N| \leq 2 \), then, for all \( q \in Q_N \), \( \Delta(\{q\}, \%) \cap A_N \neq \emptyset \).

**Proof.** There are two cases to consider.

1. Suppose \(|Q_N| = 1\). Because \( \% \in X = L(N) \), we have that \( s_N \in A_N \). Because \( s_N \in \Delta(\{s_N\}, \%) \), we have that \( \Delta(\{s_N\}, \%) \cap A_N \neq \emptyset \).

2. Suppose \(|Q_N| = 2\). Let \( q \) be the non-start state of \( N \).

   First, we show that \( \Delta(\{s_N\}, \%) \cap A_N \neq \emptyset \). If \( s_N \in A_N \), then \( s_N \in \Delta(\{s_N\}, \%) \) and \( s_N \in A_N \), and so we are done. So, suppose \( s_N \notin A_N \). Because \( \% \in L(N) \), we must have that \( s_N, \%, q \in T \) and \( q \in A_N \). Thus \( q \in \Delta(\{s_N\}, \%) \) and \( q \in A_N \), so that \( \Delta(\{s_N\}, \%) \cap A_N \neq \emptyset \).

   It remains to show that \( \Delta(\{q\}, \%) \cap A_N \neq \emptyset \). If \( q \in A_N \), then this holds, so suppose \( q \notin A_N \). Then \( s_N \in A_N \), as otherwise \( N \) would accept nothing. If \( (s_N, 0, s_N) \in T \), then \( \%(00)\% = 00 \in L(N) = X \)—contradiction. Thus \( (s_N, 0, s_N) \notin T \). Because \( 0 \in X = L(N) \), there must be a labeled path \( lp \) that is valid for \( N \), starts and ends at \( s_N \), and is labeled by \( 0 \). Since \( (s_N, 0, s_N) \notin T \), \( lp \) must visit \( q \) on its way from \( s_N \) to \( s_N \). If its first return to \( s_N \) uses label \( \% \), then \( (q, \%, s_N) \in T \), and thus \( \Delta(\{q\}, \%) \cap A_N \neq \emptyset \). Otherwise, its first return to \( s_N \) uses label \( 0 \), so that \( (q, 0, s_N) \in T \). But then \( (s_N, \%, q) \in T \), as otherwise \( lp \) wouldn’t have label \( 0 \). Thus \( s_N \in \Delta(\{s_N\}, \%), \% \), i.e., \( s_N \in \Delta(\{s_N\}, 00) \). But then \( s_N \in \Delta(\{s_N\}, 00) \), showing that \( 00 \in L(N) = X \)—contradiction. Thus \( \Delta(\{q\}, \%) \cap A_N \neq \emptyset \).

\( \square \)

**Lemma ES3.2.6**

For all \( n \in \mathbb{N} \), \( 00(0^n)11 \) is a length \( n + 4 \) element of \( X \), and, for all prefixes \( v \) of \( 00(0^n)11 \), if \( 2 \leq |v| \leq n + 3 \), then \( v \notin X \).
Proof. Suppose \( n \in \mathbb{N} \) and let \( w = 00(0^n)11 \). Clearly \( |w| = 2 + n + 2 = n + 4 \). By Lemma ES3.2.1(2), we have that \( w = 00(0^n)11 \in \{0, 1\}^* \{11\} \subseteq X \).

For the last part, suppose \( v \) is a prefix of \( w \) such that \( 2 \leq |v| \leq n + 3 \). There are two cases to consider.

- Suppose \( v = 000^i \), for some \( i \in \mathbb{N} \) such that \( i \leq n \). Because \( v = \%(00)^i \) and 11 is not a substring of \( 0^i \), it follows that \( v \not\in X \).

- Suppose \( v = 000^n1 \). Because \( v = \%(00)(0^n1) \) and 11 is not a substring of \( 0^n1 \), it follows that \( v \not\in X \).

\( \square \)

**Proposition ES3.2.7**

For all FAs \( N \), if \( L(N) = X \), then \( |Q_N| \geq |Q_M| \).

Proof. Suppose, toward a contradiction, that it is not true that, for all FAs \( N \), if \( L(N) = X \), then \( |Q_N| \geq |Q_M| \). Thus there is an FA \( N \) such that \( L(N) = X \) and \( |Q_N| < |Q_M| \). Because \( |Q_M| = 3 \), this means that \( |Q_N| \leq 2 \). Let \( n \) be two times the maximum length of the labels of \( N \)'s transitions, and let \( w = 00(0^n)11 \). By Lemma ES3.2.6, \( w \in X \) and \( |w| = n + 4 \). Because \( w \in X = L(N) \), there is a labeled path \( lp \) such that \( lp \) is valid for \( N \), the start state of \( lp \) is \( s_N \), and the label of \( lp \) is \( w \). By the definition of \( n \), and since \( |w| = n + 4 \), the labels of at least three transitions of \( lp \) must be non-\%. Let \( lp' \) be the shortest initial part of \( lp \) that contains exactly two transitions with non-\% labels, let \( v \) be the label of \( lp' \), and let \( q \) be the end state of \( lp' \). Thus \( v \) is a prefix of \( w \). By the definition of \( n \), and because exactly two of the labels of \( lp' \) are non-\%, we have that \( 2 \leq |v| \leq n \leq n + 3 \). Furthermore, by Lemma ES3.2.5, we have that \( \Delta(\{q\}, \%) \cap A_N \neq \emptyset \). Thus, there is a labeled path that is valid for \( N \), starts at \( s_N \), ends at an element of \( A_N \), and is labeled by \( v\% = v \), showing that \( v \in L(N) = X \). But by Lemma ES3.2.6, it follows that \( v \not\in X \)—contradiction. Thus, for all FAs \( N \), if \( L(N) = X \), then \( |Q_N| \geq |Q_M| \). \( \square \)