Exercise 1

(a)

(b) Define the following languages:

\[ Y = \{ w \in \{0,1\}^* \mid w \notin X \}, \]
\[ A = \{ w \in \{0,1\}^* \mid w \in X \text{ and } 0 \text{ is not a suffix of } w \}, \]
\[ B = \{ w \in \{0,1\}^* \mid w \in X \text{ and } 0, \text{ but not } 00, \text{ is a suffix of } w \}, \]
\[ C = \{ w \in \{0,1\}^* \mid w \in X \text{ and } 00 \text{ is a suffix of } w \}, \]
\[ D = \{ w \in \{0,1\}^* \mid w \notin X \text{ and } 0 \text{ is a suffix of } w \}, \]
\[ E = \{ w \in \{0,1\}^* \mid w \notin X \text{ and } 1, \text{ but not } 11, \text{ is a suffix of } w \}, \]
\[ F = \{ w \in \{0,1\}^* \mid w \notin X \text{ and } 11 \text{ is a suffix of } w \}. \]

Lemma ES4.1.1

(1) \% \in A.

(2) \{X\} \subseteq X.

(3) \{A\} \subseteq A.

(4) \{B\} \subseteq A.

(5) \{C\} \subseteq A.

(6) \{F\} \subseteq A.

(7) \{A\} \subseteq B.

(8) \{B\} \subseteq C.

(9) \{C\} \subseteq D.
We must show that \( \% \in A \). Because 0 is not a suffix of \( \% \), it remains to show that \( \% \in X \).
Suppose that \( x, y \in \{0, 1\}^* \) and \( \% = x000y \). We must show that 111 is a substring of \( y \). But
\( \% = x000y \) is impossible, and thus 111 is a substring of \( y \).

(2) Suppose \( w \in X \{1\} \). We must show that \( w \in X \). By the assumption, we have that \( w = z1 \)
for some \( z \in X \). To show that \( w \in X \), suppose \( x, y \in \{0, 1\}^* \) and \( w = x000y \). We must show
that 111 is a substring of \( y \). Because \( x000y = w = z1 \), we have that \( y = y'1 \) for some
\( y' \in \{0, 1\}^* \). Thus \( x000y'1 = x000y = z1 \), so that \( x000y' = z \). Because \( z \in X \), we have that
111 is a substring of \( y' \). Thus 111 is a substring of \( y'1 = y \).

(3) Suppose \( w \in A \{1\} \). We must show that \( w \in A \). By the assumption, we have that \( w = z1 \)
for some \( z \in A \). But \( A \subseteq X \), and thus \( w = z1 \in X \{1\} \subseteq X \), by Part (2). And 0 is not a suffix of
\( z1 = w \), completing the proof that \( w \in A \).

(4) Suppose \( w \in B \{1\} \). We must show that \( w \in A \). By the assumption, we have that \( w = z1 \)
for some \( z \in B \). But \( B \subseteq X \), and thus \( w = z1 \in X \{1\} \subseteq X \), by Part (2). And 0 is not a suffix of
\( z1 = w \), completing the proof that \( w \in A \).

(5) Suppose \( w \in C \{1\} \). We must show that \( w \in A \). By the assumption, we have that \( w = z1 \)
for some \( z \in C \). But \( C \subseteq X \), and thus \( w = z1 \in X \{1\} \subseteq X \), by Part (2). And 0 is not a suffix of
\( z1 = w \), completing the proof that \( w \in A \).

(6) Suppose \( w \in F \{1\} \). We must show that \( w \in A \). By the assumption, we have that \( w = z1 \)
for some \( z \in F \). Thus 0 is not a suffix of \( z1 = w \). So it remains to show that \( w \in X \). Suppose
\( x, y \in \{0, 1\}^* \) and \( w = x000y \). We must show that 111 is a substring of \( y \). Because \( z \in F \), we have that
111 is a suffix of \( z \). Thus \( z = z'111 \) for some \( z' \in \{0, 1\}^* \), so that \( w = z1 = z'111 \).
Since \( z'111 = w = x000y \), we have that 111 is a suffix of \( y \). Hence 111 is a substring of \( y \).

(7) Suppose \( w \in A \{0\} \). We must show that \( w \in B \). By the assumption, \( w = 0 \) for some \( z \in A \).
We have that 0 is a suffix of \( z0 = w \). Because \( z \in A \), it follows that 0 is not a suffix of \( z \).
Thus 00 is not a suffix of \( z0 = w \). So it remains to show that \( w \in X \). Suppose \( x, y \in \{0, 1\}^* \)
and \( w = x000y \). We must show that 111 is a substring of \( y \). We have that \( x000y = w = z0 \).
Because 0 is not a suffix of \( z \), there are two cases to consider.

- Suppose \( z = \% \). Thus \( x000y = z0 = 0 \)—contradiction. Thus 111 is a substring of \( y \).
Suppose $z = z'1$ for some $z' \in \{0, 1\}^*$. Thus $x000y = z0 = z'10$, so that $y = y0$ for some $y' \in \{0, 1\}^*$. Hence $x000y0 = x000y = z'10$, so that $x000y = z'1 = z$. Because $z \in A \subseteq X$, we have that 111 is a substring of $y'$. Thus 111 is a substring of $y1$. Suppose $w \in B\{0\}$. We must show that $w \in C$. By the assumption, $w = z0$ for some $z \in B$. Because $z \in B$, we have that $0 \not\in z$. Thus we have that $00$ is a suffix of $z0 = w$. So, it remains to show that $w \in X$. Suppose $x, y \in \{0, 1\}^*$ and $w = x000y$. We must show that 111 is a substring of $y$. We have that $x000y = w = z0$. Because 0, but not 00, is a suffix of $z$, there are two cases to consider.

- Suppose $z = 0$. Thus $x000y = z0 = 00$—contradiction. Thus 111 is a substring of $y$.
- Suppose $z = z'10$ for some $z' \in \{0, 1\}^*$. Thus $x000y = z0 = z'100$, so that $y = y'100$ for some $y' \in \{0, 1\}^*$. Hence $x000y100 = x000y = z'100$, so that $x000y10 = z'10 = z$. Because $z \in B \subseteq X$, we have that 111 is a substring of $y'10$. Thus 111 is a substring of $y'100 = y1$.

(8) Suppose $w \in C\{0\}$. We must show that $w \in D$. By the assumption, $w = z0$ for some $z \in C$. Thus 0 is a suffix of $z0 = w$. So, it remains to show that $w \not\in X$. Because $z \in C$, we have that 00 is a suffix of $z$. Thus $z = z'00$ for some $z' \in \{0, 1\}^*$, so that $w = z0 = z'000 = z'000\%$. Because $w = z'000\%$ and 111 is not a substring of $%$, we have that $w \not\in X$.

(9) Suppose $w \in Y\{0\}$. We must show that $w \in Y$. By the assumption, $w = z0$ for some $z \in Y$. Thus $z \not\in X$, so that there are $x, y \in \{0, 1\}^*$ such that $z = x000y$ and 111 is not a substring of $y$. Thus $w = z0 = x000y0$ and 111 is not a substring of $y0$, showing that $w \not\in X$. Thus $w \in Y$.

(10) Suppose $w \in D\{0\}$. We must show that $w \in D$. By the assumption, $w = z0$ for some $z \in D$. Because $z \in D$, we have that $z \not\in X$, so that $z \in Y$. Thus $w = z0 \in Y\{0\} \subseteq Y$, by Part (10), so that $w \not\in X$. Furthermore, 0 is a suffix of $z0 = w$, completing the proof that $w \in D$.

(11) Suppose $w \in E\{0\}$. We must show that $w \in D$. By the assumption, $w = z0$ for some $z \in E$. Because $z \in E$, we have that $z \not\in X$, so that $z \in Y$. Thus $w = z0 \in Y\{0\} \subseteq Y$, by Part (10), so that $w \not\in X$. Furthermore, 0 is a suffix of $z0 = w$, completing the proof that $w \in D$.

(12) Suppose $w \in F\{0\}$. We must show that $w \in D$. By the assumption, $w = z0$ for some $z \in F$. Because $z \in F$, we have that $z \not\in X$, so that $z \in Y$. Thus $w = z0 \in Y\{0\} \subseteq Y$, by Part (10), so that $w \not\in X$. Furthermore, 0 is a suffix of $z0 = w$, completing the proof that $w \in D$.

(13) Suppose $w \in D\{1\}$. We must show that $w \in E$. By the assumption, we have that $w = z1$ for some $z \in D$. Thus $z \not\in X$ and 0 is a suffix of $z$, so that $z = z'0$ for some $z' \in \{0, 1\}^*$. Because $w = z1 = z'01$, we have that 1, but not 11, is a suffix of $z'01 = w$. So, it remains to show that $w \not\in X$. Because $z \not\in X$, there are $x, y \in \{0, 1\}^*$ such that $z = x000y$ and 111 is not a substring of $y$. Thus $w = z1 = x000y1$. Suppose, toward a contradiction, that 111 is a substring of $y1$. Because 111 is not a substring of $y$, we have that 11 is a suffix of $y$. Thus, since $z'01 = w = x000y1$, we have that 0 = 1—contradiction. Hence 111 is not a substring of $y1$. Because $w = x000y1$ and 111 is not a substring of $y1$, we have that $w \not\in X$. 

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(15) Suppose \( w \in E\{1\} \). We must show that \( w \in F \). By the assumption, we have that \( w = z1 \) for some \( z \in E \). Thus \( z \not\in X \) and \( 1 \), but not \( 11 \), is a suffix of \( z \). Hence \( z = z'1 \) for some \( z' \in \{0,1\}^* \), and \( 1 \) is not a suffix of \( z' \). We have that \( 11 \) is a suffix of \( z'11 = z1 = w \). So, it remains to show that \( w \not\in X \). Because \( z \not\in X \), there are \( x, y \in \{0,1\}^* \) such that \( z = x000y \) and \( 111 \) is not a substring of \( y \). Hence \( w = z1 = x000y1 \). Suppose, toward a contradiction, that \( 111 \) is a substring of \( y1 \). Because \( 111 \) is not a substring of \( y \), it follows that \( 11 \) is a suffix of \( y \). Thus, since \( z'11 = w = x000y1 \), it follows that \( 1 \) is a suffix of \( z' \)—contradiction. Hence \( 111 \) is not a substring of \( y1 \). Because \( w = x000y1 \) and \( 111 \) is not a substring of \( y1 \), we have that \( w \not\in X \).

\[ \Box \]

**Lemma ES4.1.2**

For all \( w \in \{0,1\}^* \):

(A) if \( \delta(A, w) = A \), then \( w \in A \).

(B) if \( \delta(A, w) = B \), then \( w \in B \).

(C) if \( \delta(A, w) = C \), then \( w \in C \).

(D) if \( \delta(A, w) = D \), then \( w \in D \).

(E) if \( \delta(A, w) = E \), then \( w \in E \).

(F) if \( \delta(A, w) = F \), then \( w \in F \).

**Proof.** We proceed by left string induction.

(Basis Step) We must show that (A)–(F) hold, where \% has been substituted for \( w \). By Lemma ES4.1.1(1), we have that \% \( \in A \), so that Part (A) holds. And Parts (B)–(F) hold vacuously, since \( \delta(A, \%) = A \).

(Inductive Step) Suppose \( a \in \{0,1\} \) and \( w \in \{0,1\}^* \). Assume the inductive hypothesis, that Parts (A)–(F) hold. We must show that Parts (A)–(F) hold, where \( wa \) has been substituted for \( w \). There are six parts to show.

(A) Suppose \( \delta(A, wa) = A \). Since \( \delta(\delta(A, w), a) = \delta(A, wa) = A \), we have that \( (\delta(A, w), a, A) \in T \).

There are four cases to consider.

- Suppose \( \delta(A, w) = A \) and \( a = 1 \). Part (A) of the inductive hypothesis tells us that \( w \in A \). Thus \( wa = w1 \in A\{1\} \subseteq A \), by Lemma ES4.1.1(3).

- Suppose \( \delta(A, w) = B \) and \( a = 1 \). Part (B) of the inductive hypothesis tells us that \( w \in B \). Thus \( wa = w1 \in B\{1\} \subseteq A \), by Lemma ES4.1.1(4).

- Suppose \( \delta(A, w) = C \) and \( a = 1 \). Part (C) of the inductive hypothesis tells us that \( w \in C \). Thus \( wa = w1 \in C\{1\} \subseteq A \), by Lemma ES4.1.1(5).

- Suppose \( \delta(A, w) = F \) and \( a = 1 \). Part (F) of the inductive hypothesis tells us that \( w \in F \). Thus \( wa = w1 \in F\{1\} \subseteq A \), by Lemma ES4.1.1(6).
(B) Suppose \( \delta(A, wa) = B \). Since \( \delta(\delta(A, w), a) = \delta(A, wa) = B \), we have that \( (\delta(A, w), a, B) \in T \). Thus \( \delta(A, w) = A \) and \( a = 0 \). Part (A) of the inductive hypothesis tells us that \( w \in A \). Thus \( wa = w0 \in A\{0\} \subseteq B \), by Lemma ES4.1.1(7).

(C) Suppose \( \delta(A, wa) = C \). Since \( \delta(\delta(A, w), a) = \delta(A, wa) = C \), we have that \( (\delta(A, w), a, C) \in T \). Thus \( \delta(A, w) = B \) and \( a = 0 \). Part (B) of the inductive hypothesis tells us that \( w \in B \). Thus \( wa = w0 \in B\{0\} \subseteq C \), by Lemma ES4.1.1(8).

(D) Suppose \( \delta(A, wa) = D \). Since \( \delta(\delta(A, w), a) = \delta(A, wa) = D \), we have that \( (\delta(A, w), a, D) \in T \). There are four cases to consider.

- Suppose \( \delta(A, w) = C \) and \( a = 0 \). Part (C) of the inductive hypothesis tells us that \( w \in C \). Thus \( wa = w0 \in C\{0\} \subseteq D \), by Lemma ES4.1.1(9).
- Suppose \( \delta(A, w) = D \) and \( a = 0 \). Part (D) of the inductive hypothesis tells us that \( w \in D \). Thus \( wa = w0 \in D\{0\} \subseteq D \), by Lemma ES4.1.1(11).
- Suppose \( \delta(A, w) = E \) and \( a = 0 \). Part (E) of the inductive hypothesis tells us that \( w \in E \). Thus \( wa = w0 \in E\{0\} \subseteq D \), by Lemma ES4.1.1(12).
- Suppose \( \delta(A, w) = F \) and \( a = 0 \). Part (F) of the inductive hypothesis tells us that \( w \in F \). Thus \( wa = w0 \in F\{0\} \subseteq D \), by Lemma ES4.1.1(13).

(E) Suppose \( \delta(A, wa) = E \). Since \( \delta(\delta(A, w), a) = \delta(A, wa) = E \), we have that \( (\delta(A, w), a, E) \in T \). Thus \( \delta(A, w) = D \) and \( a = 1 \). Part (D) of the inductive hypothesis tells us that \( w \in D \). Thus \( wa = w1 \in D\{1\} \subseteq E \), by Lemma ES4.1.1(14).

(F) Suppose \( \delta(A, wa) = F \). Since \( \delta(\delta(A, w), a) = \delta(A, wa) = F \), we have that \( (\delta(A, w), a, F) \in T \). Thus \( \delta(A, w) = E \) and \( a = 1 \). Part (E) of the inductive hypothesis tells us that \( w \in E \). Thus \( wa = w1 \in E\{1\} \subseteq F \), by Lemma ES4.1.1(15).

\( \square \)

Now, we use the preceding lemma to show that \( L(M) = X \).

\( (L(M) \subseteq X) \) Suppose \( w \in L(M) \). Then \( w \in \{0, 1\}^* \) and \( \delta(A, w) \in \{A, B, C\} \). Thus, by Parts (A)–(C) of Lemma ES4.1.2, we have that \( w \in A \) or \( w \in B \) or \( w \in C \). But \( A \subseteq X \), \( B \subseteq X \) and \( C \subseteq X \), so that \( w \in X \).

\( (X \subseteq L(M)) \) Suppose \( w \in X \). Since \( X \subseteq \{0, 1\}^* \), we have that \( w \in \{0, 1\}^* \). Suppose, toward a contradiction, that \( w \notin L(M) \). Thus \( \delta(A, w) \in \{D, E, F\} \). Thus, by Parts (D)–(F) of Lemma ES4.1.2, we have that \( w \in D \) or \( w \in E \) or \( w \in F \). Thus \( w \notin X \)—contradiction. Thus \( w \in L(M) \).

Exercise 2

(a) The regular expression \( \alpha = ((01)^*)^* \). To see that our answer is correct, we put the description

\{states\}
A, B, C, D

\{start state\}
A

\{accepting states\}
of the FA $M_1$ in the file es4-ex2-fa. We then load $\alpha$ and $M_1$ into Forlan, convert $\alpha$ to an FA, and check that it is isomorphic to $M_1$, as follows:

```plaintext
- val reg = Reg.input "";
  @ ((01)*)*
  @ .
  val reg = - : reg
- val fa1 = FA.input "es4-ex2-fa"
  val fa1 = - : fa
- FA.isomorphic(regToFA reg, fa1);
  val it = true : bool
```

(b) $M_2$ is formed from $M_1$ by the following process. First, the state $A$ of $M_1$ is turned into the state $\langle 1, A \rangle$ of $M_2$. $\langle 1, A \rangle$ is the start state of $M_2$, since $A$ is the start state of $M_1$; it is an accepting state, since $A$ is an accepting state. Next, the state $B$ of $M_1$ is turned into the state $\langle 1, B \rangle$ of $M_2$; $\langle 1, B \rangle$ is a non-accepting state, since $B$ is a non-accepting state. Similarly, the states $C$ and $D$ of $M_1$ are turned into the non-accepting states $\langle 1, C \rangle$ and $\langle 1, D \rangle$ of $M_2$. Next, the transitions $(A, \%, B)$, $(B, \%, A)$, $(B, \%, C)$ and $(D, \%, B)$ of $M_1$ are turned into the transitions $\langle 1, A \rangle, \langle 1, B \rangle$, $\langle 1, B \rangle, \langle 1, A \rangle$, $\langle 1, B \rangle, \langle 1, C \rangle$ and $\langle 1, D \rangle, \langle 1, B \rangle$ of $M_2$, respectively. Finally, the transition $(C, 01, D)$ of $M_1$ is split into the $M_2$ transitions $\langle 1, C \rangle, 0, \langle 2, \langle C, 0, 1, D \rangle \rangle$ and $\langle 2, \langle C, 0, 1, D \rangle \rangle, 1, \langle 1, D \rangle$, where $\langle 2, \langle C, 0, 1, D \rangle \rangle$ is a new, non-accepting state of $M_2$.

Here is a drawing of $M_2$:

![Diagram of M2](image)

Continuing our Forlan session, We convert the FA $M_1$ into an EFA, as follows:

```plaintext
- val efa2 = faToEFA fa1;
  val efa2 = - : efa
- EFA.output("", efa2);
{states}
  \langle 1, A \rangle, \langle 1, B \rangle, \langle 1, C \rangle, \langle 1, D \rangle, \langle 2, \langle C, 0, 1, D \rangle \rangle
{start state}
  \langle 1, A \rangle
{accepting states}
  \langle 1, A \rangle
{transitions}
  \langle 1, A \rangle, \% -> \langle 1, B \rangle; \langle 1, B \rangle, \% -> \langle 1, A \rangle \mid \langle 1, C \rangle; \langle 1, C \rangle, 0 -> \langle 2, \langle C, 0, 1, D \rangle \rangle;
  \langle 1, D \rangle, \% -> \langle 1, B \rangle; \langle 2, \langle C, 0, 1, D \rangle \rangle, 1 -> \langle 1, D \rangle
```
It is easy to check that $M_2$ is the outputted EFA.

(c) Continuing with our Forlan session, we rename the states of $M_2$, producing an EFA $M_3$, as follows:

- val efa3 = EFA.renameStatesCanonically efa2;
- val efa3 = - : efa
- EFA.output("", efa3);

{states}
A, B, C, D, E
{start state}
A
{accepting states}
A
{transitions}
A, % -> B; B, % -> A | C; C, 0 -> E; D, % -> B; E, 1 -> D
val it = () : unit

Here is a drawing of $M_3$:

(d) The NFA $M_4$ is formed from $M_3$ by the following process. Its states and start state are the same as those of $M_3$. The accepting states of $M_4$ are the elements of emptyCloseBackwards($A_{M_3}$) = emptyCloseBackwards({A}) = {A, B, D}. The transitions of $M_4$ are formed by processing the non-% transitions of $M_3$. The transition (C,0,E) of $M_3$ is turned into the transitions (A,0,E), (B,0,E), (C,0,E) and (D,0,E), since emptyCloseBackwards({C}) = {A, B, C, D} and emptyClose({E}) = {E}. The transition (E,1,D) of $M_3$ is turned into the transitions (E,1,A), (E,1,B), (E,1,C) and (E,1,D), since emptyCloseBackwards({E}) = {E} and emptyClose({D}) = {A, B, C, D}.

Here is a drawing of $M_4$:
Continuing our Forlan session, we convert the EFA $M_3$ into an NFA, and output the result, as follows.

```haskell
val nfa4 = efaToNFA efa3;
val nfa4 = - : nfa
val it = () : unit
```

We form the DFA $\{states\}$

- $A$, $B$, $C$, $D$, $E$
- $\{accepting states\}$
- $\{transitions\}$
- $\{start state\}$
- $\{\langle\rangle\}$

It is easy to check that $M_4$ is the outputted NFA.

(e) We form the DFA $M_5$ from the NFA $M_4$, by the following process, which involves the construction of a set $X$ of sets of states of $M_4$. First, $\{A\}$ is added to $X$ and $\langle A \rangle$ is made the start state of $M_5$, since $A$ is $M_4$'s start state. Since $A$ is an accepting state of $M_4$, $\langle A \rangle$ is an accepting state of $M_5$.

Since $\{A\} \in X$ and $\Delta(\{A\}, 0) = \{E\}$, we add $\{E\}$ to $X$, $\langle E \rangle$ to $Q_{M_5}$ and $\langle\langle A\rangle, 0, \langle E \rangle\rangle$ to $T_{M_5}$. Since $\{A\} \in X$ and $\Delta(\{A\}, 1) = \emptyset$, we add $\emptyset$ to $X$, $\langle\emptyset\rangle$ to $Q_{M_5}$ and $\langle\langle A\rangle, 1, \langle\emptyset\rangle\rangle$ to $T_{M_5}$.

Since $\{E\} \in X$ and $\Delta(\{E\}, 0) = \emptyset$, we add $\emptyset$ to $X$, $\langle\emptyset\rangle$ to $Q_{M_5}$ and $\langle\langle E\rangle, 0, \langle\emptyset\rangle\rangle$ to $T_{M_5}$. Since $\{E\} \in X$ and $\Delta(\{E\}, 1) = \{A, B, C, D\}$, we add $\{A, B, C, D\}$ to $X$, $\langle A, B, C, D \rangle$ to $Q_{M_5}$ and $\langle\langle E\rangle, 1, \langle A, B, C, D\rangle\rangle$ to $T_{M_5}$. Since $A$ is an accepting state of $M_5$, $\langle A, B, C, D \rangle$ is an accepting state of $M_5$.

Since $\emptyset \in X$ and $\Delta(\emptyset, 0) = \emptyset$, we add $\emptyset$ to $X$, $\langle\emptyset\rangle$ to $Q_{M_5}$ and $\langle\langle\emptyset\rangle, 0, \langle\emptyset\rangle\rangle$ to $T_{M_5}$. Since $\emptyset \in X$ and $\Delta(\emptyset, 1) = \emptyset$, we add $\emptyset$ to $X$, $\langle\emptyset\rangle$ to $Q_{M_5}$ and $\langle\langle\emptyset\rangle, 1, \langle\emptyset\rangle\rangle$ to $T_{M_5}$.

Here is a drawing of $M_5$:

![Diagram of M5]

Continuing our Forlan session, we convert our NFA $M_4$ into a DFA, as follows:

```haskell
val dfa5 = nfaToDFA nfa4;
val dfa5 = - : dfa
val it = () : unit
```

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It is easy to check that $M_5$ is the outputted DFA.

(f) Continuing our Fornan session, we rename the states of $M_5$, producing a DFA $M_6$, as follows:

```ml
val it = () : unit
```

Here is a drawing of $M_6$: