CIS 570 — Introduction to Formal Language Theory — Fall 2007

Exercise Set 1

Due by 4:00 p.m. on Tuesday, September 18

Exercise 1 (20 points)

Define a function $f \in \mathbb{N} \to \mathbb{N}$ by recursion:

$$\begin{aligned} f(0) &= 0, \\ f(n+1) &= f(n) + n, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Use mathematical induction to show that, for all $n \in \mathbb{N}$,

$$f(n) = \frac{n^2 - n}{2}.$$

Exercise 2 (20 points)

Let

$$a = \frac{1+\sqrt{5}}{2}, \qquad b = \frac{1-\sqrt{5}}{2} \qquad \text{and} \qquad c = \frac{1}{\sqrt{5}}.$$

Define a function $f \in \mathbb{N} \to \mathbb{N}$ by strong recursion: for all $n \in \mathbb{N}$,

$$f(n) = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ f(n-1) + f(n-2), & \text{if } n \geq 2. \end{cases}$$

Use strong induction to prove that, for all $n \in \mathbb{N}$, $f(n) = c(a^n - b^n)$.

Exercise 3 (20 points)

(a) Either prove or disprove the following statement:

For all sets A and B,

$$A = (A \cap B) \cup (A - B).$$

[10 points]

(b) Either prove or disprove the following statement:

For all sets A, B and C,

$$(A - B) - C = A - (B - C).$$

[10 points]

Exercise 4 (40 points)

Define a function $\mathbf{diff} \in \{0,1\}^* \to \mathbb{Z}$ by: for all $w \in \{0,1\}^*$,

$$\mathbf{diff}(w) = \text{the number of 1's in } w - \text{the number of 0's in } w.$$

Thus:

- $\mathbf{diff}(\%) = 0$;
- $\mathbf{diff}(0) = -1$;
- $\mathbf{diff}(1) = 1$;
- for all $x, y \in \{0,1\}^*$, $\mathbf{diff}(xy) = \mathbf{diff}(x) + \mathbf{diff}(y)$.

And, for all $w \in \{0,1\}^*$, $\mathbf{diff}(w) = 0$ iff w has an equal number of 1's and 0's.

Let X be the least subset of $\{0,1\}^*$ such that:

(1) $1 \in X$;

(2) for all $x, y \in X$, $x0y \in X$;

(3) for all $x, y \in X$, $0xy \in X$.

Let $Y = \{\, w \in \{0,1\}^* \mid \mathbf{diff}(w) = 1 \text{ and, for all prefixes } v \text{ of } w, \mathbf{diff}(v) \leq 1 \,\}$.

(a) Prove that $X \subseteq Y$. Hint: use induction on X. [15 points]

(b) Prove that $Y \subseteq X$, completing the proof that $X = Y$. Hint: use strong string induction. [25 points]