Exercise 1 (20 points)

Define a function \( f \in \mathbb{N} \rightarrow \mathbb{N} \) by recursion:

\[
f(0) = 0, \\
f(n + 1) = f(n) + n, \text{ for all } n \in \mathbb{N}.
\]

Use mathematical induction to show that, for all \( n \in \mathbb{N} \),

\[
f(n) = \frac{n^2 - n}{2}.
\]

Exercise 2 (20 points)

Let \( a = \frac{1 + \sqrt{5}}{2}, \ b = \frac{1 - \sqrt{5}}{2} \) and \( c = \frac{1}{\sqrt{5}} \).

Define a function \( f \in \mathbb{N} \rightarrow \mathbb{N} \) by strong recursion: for all \( n \in \mathbb{N} \),

\[
f(n) = \begin{cases} 
0, & \text{if } n = 0, \\
1, & \text{if } n = 1, \\
f(n - 1) + f(n - 2), & \text{if } n \geq 2.
\end{cases}
\]

Use strong induction to prove that, for all \( n \in \mathbb{N} \), \( f(n) = c(a^n - b^n) \).

Exercise 3 (20 points)

(a) Either prove or disprove the following statement:

For all sets \( A \) and \( B \),

\[
A = (A \cap B) \cup (A - B).
\]

[10 points]

(b) Either prove or disprove the following statement:

For all sets \( A, B \) and \( C \),

\[
(A - B) - C = A - (B - C).
\]

[10 points]
Exercise 4 (40 points)

Define a function $\text{diff} \in \{0, 1\}^* \rightarrow \mathbb{Z}$ by: for all $w \in \{0, 1\}^*$,

$$\text{diff}(w) = \text{the number of 1's in } w - \text{the number of 0's in } w.$$ 

Thus:

- $\text{diff}(\%) = 0$;
- $\text{diff}(0) = -1$;
- $\text{diff}(1) = 1$;
- for all $x, y \in \{0, 1\}^*$, $\text{diff}(xy) = \text{diff}(x) + \text{diff}(y)$.

And, for all $w \in \{0, 1\}^*$, $\text{diff}(w) = 0$ iff $w$ has an equal number of 1’s and 0’s.

Let $X$ be the least subset of $\{0, 1\}^*$ such that:

1. $1 \in X$;
2. for all $x, y \in X$, $x0y \in X$;
3. for all $x, y \in X$, $0xy \in X$.

Let $Y = \{ w \in \{0, 1\}^* \mid \text{diff}(w) = 1 \text{ and, for all prefixes } v \text{ of } w, \text{diff}(v) \leq 1 \}$.

(a) Prove that $X \subseteq Y$. Hint: use induction on $X$. [15 points]

(b) Prove that $Y \subseteq X$, completing the proof that $X = Y$. Hint: use strong string induction. [25 points]