Exercise Set 2

Due by 4:00 p.m. on Tuesday, October 9

Exercise 1 (20 points)

Let $X = \{ w \in \{0,1\}^* \mid |w| \leq 4$ and neither 000 nor 111 is a substring of $w \}$. Use Forlan to find and show the correctness of a regular expression $\alpha$ such that $L(\alpha) = X$. Try to minimize the size of $\alpha$, and use Forlan to display the size of $\alpha$. Try to do as much as possible of the work of finding and showing the correctness of $\alpha$ using Forlan. (Include a listing of your Forlan session.)

Exercise 2 (20 points)

(a) Prove that, for all $n \in \mathbb{N}$ and $A, B \in \text{Lan}$, if $n \geq 1$ and $A^n \subseteq B$, then $A^{n+1}A^* \cup B = A^nA^* \cup B$. [15 points]

(b) Prove that, for all $n \in \mathbb{N}$ and $\alpha, \beta \in \text{Reg}$, if $n \geq 1$ and $L(\alpha^n) \subseteq L(\beta)$, then $\alpha^{n+1}\alpha^* + \beta \approx \alpha^n\alpha^* + \beta$. [5 points]

Exercise 3 (35 points)

Define a function $\text{diff} \in \{0,1\}^* \rightarrow \mathbb{Z}$ by: for all $w \in \{0,1\}^*$,

$$\text{diff}(w) = \text{the number of 1's in } w - \text{the number of 0's in } w.$$ 

Thus:

- $\text{diff}(\%) = 0$;
- $\text{diff}(0) = -1$;
- $\text{diff}(1) = 1$;
- for all $x, y \in \{0,1\}^*$, $\text{diff}(xy) = \text{diff}(x) + \text{diff}(y)$.

Let $X = \{ w \in \{0,1\}^* \mid \text{for all prefixes } v \text{ of } w, 0 \leq \text{diff}(v) \leq 2 \}$.

(a) Find a regular expression $\alpha$ such that $L(\alpha) = X$. [10 points]

(b) Prove that your answer to Part (a) is correct. [25 points]
Exercise 4 (25 points)

Consider the model answer to Exercise 4(b) of Exercise Set 1. Turn the proof that \( Y \subseteq X \) into the definition of a Standard ML/Forlan function

```ml
val expl : int * str -> unit
```

that, given an indentation level and an element \( w \in Y \), prints out, at that indentation level, an explanation of why \( w \in X \).

As closely as possible, make the structure of your function definition match the structure of the proof. In particular: induction in the proof should correspond to recursion in your function definition; division into cases in the proof should correspond to the use of conditionals/pattern matching in the function definition; and the use of the lemmas in the proof should correspond to the use of auxiliary functions in the function definition.

Then, use your function to finish writing a program whose main function

```ml
val explain : unit -> unit
```

reads (using `Str.input`) a string \( w \) from the standard input, issues an error message, if \( w \not\in Y \), and uses `expl` to explain why \( w \in X \), otherwise.

For example:

- if your program is given the string `100011101`, then it should output the following explanation for why this string is in \( X \):

  \[
  100011101 = 1 \otimes 0 \otimes 0011101 \, \text{is in} \, X, \, \text{by (2)} \\
  1 \, \text{is in} \, X, \, \text{by (1)} \\
  0011101 = 0 \otimes 011 \otimes 101 \, \text{is in} \, X, \, \text{by (3)} \\
  011 = 0 \otimes 1 \otimes 1 \, \text{is in} \, X, \, \text{by (3)} \\
  1 \, \text{is in} \, X, \, \text{by (1)} \\
  1 \, \text{is in} \, X, \, \text{by (1)} \\
  101 = 1 \otimes 0 \otimes 1 \, \text{is in} \, X, \, \text{by (2)} \\
  1 \, \text{is in} \, X, \, \text{by (1)} \\
  1 \, \text{is in} \, X, \, \text{by (1)}
  \]

- if your program is given the string `0110`, then it should explain why this string isn’t in \( Y \), saying:

  \[
  \text{diff of string is 0 not 1}
  \]

- if your program is given the string `01110`, then it should explain why this string isn’t in \( Y \), saying:

  \[
  \text{prefix 0111 of string has diff 2 which is greater-than 1}
  \]
• if your program is given the string 1020, then it should explain why this string isn’t in \( Y \), saying:

\[
\text{string has symbol other than 0/1 : 2}
\]

Supply a listing of your program, as well as a transcript showing how you tested your program. Keep an electronic copy of your program, in case you are asked to make it available for further testing.

Hint: you may adapt the SML/Forlan program solving Exercise 4 of last year’s Exercise Set 2. The Forlan WWW site contains links to documentation on Standard ML and a useful comparison of OCaml and Standard ML.