Exercise 1 (70 points)

Let
\[ X = \{ 0^i 1^j 2^k 3^l \mid i, j, k, l \in \mathbb{N} \text{ and } i + j = k + l \}. \]

(a) Find a grammar \( G \) such that \( L(G) = X \). [20 points]

(b) Find parse trees that are valid for \( G \), whose root labels are \( s_G \) and whose yields are the strings 00123 and 00112333. Use Forlan to check that your answers are correct. (Include a transcript of your Forlan session.) [5 points]

(c) Use Forlan to provide some additional evidence that \( L(G) = X \), making use of some test cases that are in \( X \), as well as some that are not in \( X \). (Include a transcript of your Forlan session.) [10 points]

(d) Prove that your answer to Part (a) is correct. [20 points]

(e) Prove that \( X \) is not regular. [15 points]
Exercise 2 (30 points)

Define a function \( \text{diff} \in \{0, 1\}^* \rightarrow \mathbb{Z} \) by: for all \( w \in \{0, 1\}^* \),

\[
\text{diff}(w) = \text{the number of 1's in } w - \text{the number of 0's in } w.
\]

Thus:

- \( \text{diff}(\%) = 0 \);
- \( \text{diff}(0) = -1 \);
- \( \text{diff}(1) = 1 \);
- for all \( x, y \in \{0, 1\}^* \), \( \text{diff}(xy) = \text{diff}(x) + \text{diff}(y) \).

Define \( \text{AllSubGood} \in \mathbb{N} \rightarrow \text{Lan} \) by: for all \( n \in \mathbb{N} \), \( \text{AllSubGood}(n) = \{ w \in \{0, 1\}^* \mid \text{for all substrings } v \text{ of } w, |\text{diff}(v)| \leq n \} \).

(a) Define a Forlan/SML function

\[
\text{val allSubGoodDFA : int -> dfa}
\]

such that, for all \( n \in \mathbb{N} \), \( \text{allSubGoodDFA } n \) is a DFA with as few states as possible that accepts \( \text{AllSubGood}(n) \). (You don’t need to worry about what \( \text{allSubGoodDFA} \) does when called with a negative number.) Carefully document your code. [20 points]

(b) Use your \( \text{allSubGoodDFA} \) function to generate and display (using DFA.output) DFAs accepting the languages \( \text{AllSubGood}(2) \) and \( \text{AllSubGood}(3) \). Draw each of these machines as clearly as possible. (Include a transcript of your Forlan session.) [10 points]