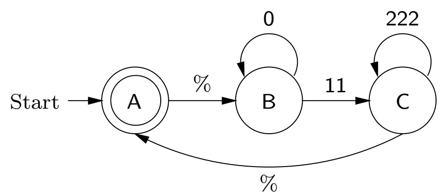
CIS 570 — Introduction to Formal Language Theory — Fall 2007

Mid-term Examination

Model Answers

Question 1



Question 2

(The parenthetical remarks aren't required.)

(a) false (because 00011 is in X, but is not in Y);

(b) true (because if every occurrence of 00 is immediately followed by an occurrence of 11, then it is also eventually followed by an occurrence of 11; and 00011 is in X, but is not in Y);

(c) false (because 00011 is in X, but is not in Y);

(d) false (because % is in both X and Y);

(e) true (because % is in both X and Y).

Question 3

(a)

$$(01^*01^*)^*$$

(b) Let $A = \{0\}\{1\}^*\{0\}\{1\}^*$. Because $L(\alpha) = A^*$, it will suffice to show that $A^* = X$. We show that $A^* \subseteq X \subseteq A^*$.

($A^* \subseteq X$) It will suffice to show that, for all $n \in \mathbb{N}$, $A^n \subseteq X$. We proceed by mathematical induction.

(Basis Step) Because $\mathbf{zeros}(\%) = 0$ is even, and 1 is not a prefix of %, we have that $\% \in X$. Hence $A^0 = \{\%\} \subseteq X$.

(Inductive Step) Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: $A^n \subseteq X$. We must show that $A^{n+1} \subseteq X$. We have that

$$\begin{aligned} A^{n+1} &= AA^n \\ &\subseteq AX \qquad \text{(inductive hypothesis)}. \end{aligned}$$

Thus, it will suffice to show that $AX \subseteq X$. Suppose $w \in AX$. We must show that $w \in X$. Because $w \in AX$, we have that $w = xy$ for some $x \in A$ and $y \in X$. Since $x \in A$, we have that $x = \mathsf{0}\mathsf{1}^n\mathsf{0}\mathsf{1}^m$ for some $n, m \in \mathbb{N}$. Hence $\mathbf{zeros}(x) = 2$. Because $y \in X$, we have that $\mathbf{zeros}(y)$ is even. Thus $\mathbf{zeros}(w) = \mathbf{zeros}(x) + \mathbf{zeros}(y)$ is even. Furthermore, $\mathsf{0}$ is a prefix of x, and so is a prefix of $xy = w$, showing that $\mathsf{1}$ is not a prefix of w. Thus $w \in X$.

$(X \subseteq A^*)$ Because $X \subseteq \{\mathsf{0},\mathsf{1}\}^*$, it is sufficient to show that, for all $w \in \{\mathsf{0},\mathsf{1}\}^*$,

$$\text{if } w \in X, \text{ then } w \in A^*.$$

We proceed by strong string induction. Suppose $w \in \{\mathsf{0},\mathsf{1}\}^*$, and assume the inductive hypothesis: for all $x \in \{\mathsf{0},\mathsf{1}\}^*$, if x is a proper substring of w, then

$$\text{if } x \in X, \text{ then } x \in A^*.$$

We must show that

$$\text{if } w \in X, \text{ then } w \in A^*.$$

Suppose $w \in X$. We must show that $w \in A^*$. There are three cases to consider.

- Suppose $w = \%$. Then $w = \% \in A^*$.
- Suppose $w = \mathsf{0}x$ for some $x \in \{\mathsf{0},\mathsf{1}\}^*$. Let $n \in \mathbb{N}$ be greatest such that $\mathsf{1}^n$ is a prefix of x (n can always be 0, as $\mathsf{1}^0 = \%$ is a prefix of x), and let $y \in \{\mathsf{0},\mathsf{1}\}^*$ be such that $x = \mathsf{1}^n y$. Hence $\mathsf{1}$ is not a prefix of y (otherwise $\mathsf{1}^{n+1}$ is a prefix of x), and $w = \mathsf{0}\mathsf{1}^n y$.

  Suppose, toward a contradiction, that $y = \%$. Thus $w = \mathsf{0}\mathsf{1}^n$, so that $\mathbf{zeros}(w) = 1$. But $w \in X$, and so $\mathbf{zeros}(w)$ is even—contradiction. Thus $y \neq \%$. But $\mathsf{1}$ is not a prefix of y, and thus $y = \mathsf{0}z$, for some $z \in \{\mathsf{0},\mathsf{1}\}^*$, so that $w = \mathsf{0}\mathsf{1}^n\mathsf{0}z$.

  Let $m \in \mathbb{N}$ be greatest such that $\mathsf{1}^m$ is a prefix of z, and let $u \in \{\mathsf{0},\mathsf{1}\}^*$ be such that $z = \mathsf{1}^m u$. Hence $\mathsf{1}$ is not a prefix of u, and $w = \mathsf{0}\mathsf{1}^n\mathsf{0}\mathsf{1}^m u$. Because $w \in X$, we have that $\mathbf{zeros}(w)$ is even. And $\mathbf{zeros}(\mathsf{0}\mathsf{1}^n\mathsf{0}\mathsf{1}^m) = 2$, so that $\mathbf{zeros}(u)$ is even. Thus $u \in X$. Because u is a proper substring of w, the inductive hypothesis tells us that $u \in A^*$. Hence $w = (\mathsf{0}\mathsf{1}^n\mathsf{0}\mathsf{1}^m)u \in AA^* \subseteq A^*$.
- Suppose $w = \mathsf{1}x$ for some $x \in \{\mathsf{0},\mathsf{1}\}^*$. Thus $\mathsf{1}$ is a prefix of w, contradicting the fact that $w \in X$. Hence $w \in A^*$.