Question 1 (25 points)

Find a finite automaton \( M \) such that \( L(M) = (\{0\}^* \{1\}\{22\}^*)^* \).

Question 2 (15 points)

Let
\[
X = \{ w \in \{0,1\}^* \mid \text{for all } x, y \in \{0,1\}^*, \text{ if } w = x00y, \text{ then } 11 \text{ is a substring of } y \},
\]
\[
Y = \{ w \in \{0,1\}^* \mid \text{for all } x, y \in \{0,1\}^*, \text{ if } w = x00y, \text{ then } 11 \text{ is a prefix of } y \}.
\]

For each of the following five statements, say whether the statement is true or false. You don't have to prove that your claims are correct.

(a) \( X \subset Y \) (\( X \) is a proper subset of \( Y \)); [3 points]
(b) \( Y \subset X \) (\( Y \) is a proper subset of \( X \)); [3 points]
(c) \( X = Y \); [3 points]
(d) \( X \cap Y = \emptyset \); [3 points]
(e) \( X \cap Y \neq \emptyset \). [3 points]

Question 3 (60 points)

Given \( w \in \{0,1\}^* \), we write \( \text{zeros}(w) \) for the number of occurrences of 0 in \( w \). For example, \( \text{zeros}(01101) = 2 \). Let \( X = \{ w \in \{0,1\}^* \mid \text{zeros}(w) \text{ is even and } 1 \text{ is not a prefix of } w \} \). For example, 01101 \( \in X \), but 111 \( \notin X \), even though \( \text{zeros}(111) = 0 \) is even.

(a) Find a regular expression \( \alpha \) such that \( L(\alpha) = X \). [20 points]

(b) Prove that your answer to Part (a) is correct. [40 points]