Exercise 1

As usual, we say that a term \( t \) is a normal form iff there is no term \( t' \) such that \( t \rightarrow t' \).

We begin by proving two lemmas.

Lemma 1.1

For all numeric values \( nv \), \( nv \) is a normal form.

Proof. Define a predicate \( P \) on numeric values by: \( P(nv) \) iff \( nv \) is a normal form. We use the principle of structural induction on numeric values to prove that, for all numeric values \( nv \), \( P(nv) \).

(Zero) We must show \( P(0) \), i.e., that \( 0 \) is a normal form. This follows by the inversion lemma for the evaluation relation, because none of the evaluation rules apply to \( 0 \).

(Successor) Suppose \( nv \) is a numeric value, and assume the inductive hypothesis, \( P(nv) \), i.e., \( nv \) is a normal form. We must show \( P(succ \: nv) \), i.e., \( succ \: nv \) is a normal form. Suppose, toward a contradiction, that \( succ \: nv \) is not a normal form. Thus \( succ \: nv \rightarrow t \) for some term \( t \). Applying the inversion lemma for the evaluation relation, only case (Succ) could apply, so that \( t = succ \: t' \), for some \( t' \) such that \( nv \rightarrow t' \). But this contradicts the fact that \( nv \) is a normal form. Thus we have that \( succ \: nv \) is a normal form.

Lemma 1.2

For all answers \( a \), \( a \) is a normal form.

Proof. If \( a \) is true, false or error, the inversion lemma for the evaluation relation shows that \( a \) is a normal form. Otherwise, \( a \) is a numeric value, and the result follows by Lemma 1.1.

Now we use our lemmas to prove the exercise’s result. Define a predicate \( P \) on pairs of terms by: \( P(t_1, t_2) \) iff, for all terms \( t'_2 \), if \( t_1 \rightarrow t'_2 \), then \( t_2 = t'_2 \). It will suffice to show that, for all terms \( t_1 \) and \( t_2 \), if \( t_1 \rightarrow t_2 \), then \( P(t_1, t_2) \). (To see that this is so, suppose \( t_1, t_2 \) and \( t'_2 \) are terms, \( t_1 \rightarrow t_2 \) and \( t_1 \rightarrow t'_2 \). Then \( P(t_1, t_2) \), so that \( t_2 = t'_2 \).) We proceed using the principle of induction on the evaluation relation.

(IffTrue) Suppose \( t_2 \) and \( t_3 \) are terms. We must show that \( P(\text{if } \text{true } \text{then } t_2 \text{ else } t_3, t_2) \). Suppose \( t \) is a term and if \( \text{true } \text{then } t_2 \text{ else } t_3 \rightarrow t \). We must show that \( t_2 = t \). By Lemma 1.2, we have that \( \text{true} \) is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (IffTrue) can apply, so that \( t = t_2 \), i.e., \( t_2 = t \).

(IffFalse) Similar to (IffTrue).
Suppose \( n v \) is a numeric value and \( t_2 \) and \( t_3 \) are terms. We must show that \( P(\text{if } n v \text{ then } t_2 \text{ else } t_3, \text{error}) \). Suppose \( t \) is a term and if \( n v \) then \( t_2 \) else \( t_3 \) \( \rightarrow t \). We must show that \( \text{error} = t \). By Lemma 1.2, we have that \( n v \) is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (IfNum) can apply, so that \( t = \text{error} \), i.e., \( \text{error} = t \).

(IfNum) Similar to (IfNum).

(If) Suppose \( t_1, t'_1, t_2 \) and \( t_3 \) are terms, \( t_1 \rightarrow t'_1 \), and assume the inductive hypothesis, \( P(t_1, t'_1) \). We must show that \( P(\text{if } t_1 \text{ then } t_2 \text{ else } t_3, \text{if } t'_1 \text{ then } t_2 \text{ else } t_3) \). Suppose \( t \) is a term and if \( t_1 \) then \( t_2 \) else \( t_3 \) \( \rightarrow t \). We must show that if \( t'_1 \) then \( t_2 \) else \( t_3 = t \). By Lemma 1.2, because \( t_1 \) is not a normal form, we have that \( t_1 \) is not an answer. Thus, applying the inversion lemma for the evaluation relation, only case (If) can apply, so that \( t = \text{if } t'_1 \text{ then } t_2 \text{ else } t_3 \), for some term \( t''_1 \) such that \( t_1 \rightarrow t''_1 \). By the inductive hypothesis, it follows that \( t'_1 = t''_1 \). Thus if \( t'_1 \) then \( t_2 \) else \( t_3 = t''_1 \) then \( t_2 \) else \( t_3 = t \).

(SuccBool) Suppose \( b v \) is a boolean value. We must show that \( P(\text{succ } b v, \text{error}) \). Suppose \( t \) is a term and \( \text{succ } b v \rightarrow t \). We must show that \( \text{error} = t \). By Lemma 1.2, we have that \( b v \) is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (SuccBool) can apply, so that \( t = \text{error} \), i.e., \( \text{error} = t \).

(SuccError) We must show that \( P(\text{succ } \text{error}, \text{error}) \). Suppose \( t \) is a term and \( \text{succ } \text{error} \rightarrow t \). We must show that \( \text{error} = t \). By Lemma 1.2, we have that \( \text{error} \) is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (SuccError) can apply, so that \( t = \text{error} \), i.e., \( \text{error} = t \).

(Succ) Suppose \( t_1 \) and \( t'_1 \) are terms, \( t_1 \rightarrow t'_1 \) and assume the inductive hypothesis, \( P(t_1, t'_1) \). We must show that \( P(\text{succ } t_1, \text{succ } t'_1) \). Suppose \( t \) is a term and \( \text{succ } t_1 \rightarrow t \). We must show that \( \text{succ } t'_1 = t \). By Lemma 1.2, because \( t_1 \) is not a normal form, we have that \( t_1 \) is not an answer. Thus, applying the inversion lemma for the evaluation relation, only case (Succ) can apply, so that \( t = \text{succ } t'_1 \), for some term \( t''_1 \) such that \( t_1 \rightarrow t''_1 \). By the inductive hypothesis, it follows that \( t'_1 = t''_1 \). Thus \( \text{succ } t'_1 = \text{succ } t''_1 = t \).

(PredBool) Suppose \( b v \) is a boolean value. We must show that \( P(\text{pred } b v, \text{error}) \). Suppose \( t \) is a term and \( \text{pred } b v \rightarrow t \). We must show that \( \text{error} = t \). By Lemma 1.2, we have that \( b v \) is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (PredBool) can apply, so that \( t = \text{error} \), i.e., \( \text{error} = t \).

(PredZero) We must show that \( P(\text{pred } 0, \text{error}) \). Suppose \( t \) is a term and \( \text{pred } 0 \rightarrow t \). We must show that \( \text{error} = t \). By Lemma 1.2, we have that \( 0 \) is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (PredZero) can apply, so that \( t = \text{error} \), i.e., \( \text{error} = t \).

(PredSucc) Suppose \( n v \) is a numeric value. We must show that \( P(\text{pred}(\text{succ } n v), n v) \). Suppose \( t \) is a term and \( \text{pred}(\text{succ } n v) \rightarrow t \). We must show that \( n v = t \). By Lemma 1.2, we have that \( \text{succ } n v \) is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (PredSucc) can apply, so that \( t = n v \), i.e., \( n v = t \).
(PredError) We must show that $P(\text{pred error}, \text{error})$. Suppose $t$ is a term and $\text{pred error} \to t$. We must show that $\text{error} = t$. By Lemma 1.2, we have that $\text{error}$ is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (PredError) can apply, so that $t = \text{error}$, i.e., $\text{error} = t$.

(Pred) Suppose $t_1$ and $t'_1$ are terms, $t_1 \to t'_1$ and assume the inductive hypothesis, $P(t_1, t'_1)$. We must show that $P(\text{pred } t_1, \text{pred } t'_1)$. Suppose $t$ is a term and $\text{pred } t_1 \to t$. We must show that $\text{pred } t'_1 = t$. By Lemma 1.2, because $t_1$ is not a normal form, we have that $t_1$ is not an answer. Thus, applying the inversion lemma for the evaluation relation, only case (Pred) can apply, so that $t = \text{pred } t'_1$, for some term $t''_1$ such that $t_1 \to t''_1$. By the inductive hypothesis, it follows that $t'_1 = t''_1$. Thus $\text{pred } t'_1 = \text{pred } t''_1 = t$.

(IszeroBool) Similar to (PredBool).

(IszeroZero) We must show that $P(\text{iszero } 0, \text{true})$. Suppose $t$ is a term and $\text{iszero } 0 \to t$. We must show that $\text{true} = t$. By Lemma 1.2, we have that 0 is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (IszeroZero) can apply, so that $t = \text{true}$, i.e., $\text{true} = t$.

(IszeroSucc) Suppose $nv$ is a numeric value. We must show that $P(\text{iszero } (\text{succ } nv), \text{false})$. Suppose $t$ is a term and $\text{iszero } (\text{succ } nv) \to t$. We must show that $\text{false} = t$. By Lemma 1.2, we have that $\text{succ } nv$ is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (IszeroSucc) can apply, so that $t = \text{false}$, i.e., $\text{false} = t$.

(IszeroError) Similar to (PredError).

(Iszero) Similar to (Pred).

(OtherwiseValue) Suppose $t$ is a term and $v$ is a value. We must show that $P(v \text{ otherwise } t, v)$. Suppose $t'$ is a term and $v \text{ otherwise } t \to t'$. We must show that $v = t'$. By Lemma 1.2, we have that $v$ is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (OtherwiseValue) can apply, so that $t' = v$, i.e., $v = t'$.

(OtherwiseError) Suppose $t$ is a term. We must show that $P(\text{error otherwise } t, t)$. Suppose $t'$ is a term and $\text{error otherwise } t \to t'$. We must show that $t = t'$. By Lemma 1.2, we have that $\text{error}$ is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (OtherwiseError) can apply, so that $t' = t$, i.e., $t = t'$.

(Otherwise) Suppose $t_1$, $t_2$ and $t'_1$ are terms, $t_1 \to t'_1$, and assume the inductive hypothesis, $P(t_1, t'_1)$. We must show that $P(t_1 \text{ otherwise } t_2, t'_1 \text{ otherwise } t_2)$. Suppose $t$ is a term and $t_1 \text{ otherwise } t_2 \to t$. We must show that $t'_1 \text{ otherwise } t_2 = t$. By Lemma 1.2, because $t_1$ is not a normal form, we have that $t_1$ is not an answer. Thus, applying the inversion lemma for the evaluation relation, only case (Otherwise) can apply, so that $t = t''_1 \text{ otherwise } t_2$, for some term $t''_1$ such that $t_1 \to t''_1$. By the inductive hypothesis, it follows that $t'_1 = t''_1$. Thus $t'_1 \text{ otherwise } t_2 = t''_1 \text{ otherwise } t_2 = t$. 

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Exercise 2

Define a predicate $P$ on pairs of terms and answers by: $P(t, a)$ iff, for all answers $a'$, if $t \Rightarrow a'$, then $a = a'$. It will suffice show that, for all terms $t$ and answers $a$, if $t \Rightarrow a$, then $P(t, a)$. (To see that this is so, suppose $t$ is a term, $a$ and $a'$ are answers, $t \Rightarrow a$ and $t \Rightarrow a'$. Then $P(t, a)$, so that $a = a'$.)

We proceed using the principle of induction on the complete evaluation relation.

(True) We must show that $P(\text{true}, \text{true})$. Suppose $a$ is an answer and true $\Rightarrow a$. We must show that $\text{true} = a$. Applying the inversion lemma for the complete evaluation relation, only case (True) can apply, so that $a = \text{true}$, i.e., $\text{true} = a$.

(False) Similar to (True).

(Zero) Similar to (True).

(Error) Similar to (True).

(IfTrue) Suppose $t_1$, $t_2$ and $t_3$ are terms, $a$ is an answer, $t_1 \Rightarrow \text{true}$ and $t_2 \Rightarrow a$, and assume the inductive hypothesis, $P(t_1, \text{true})$ and $P(t_2, a)$. We must show that $P(\text{if } t_1 \text{ then } t_2 \text{ else } t_3, a)$. Suppose $a'$ is an answer and if $t_1$ then $t_2$ else $t_3 \Rightarrow a'$. We must show that $a = a'$. Because $P(t_1, \text{true})$, it follows that $t_1$ doesn’t completely evaluate to false, a numeric value or error. Thus, applying the inversion lemma for the complete evaluation relation, only case (IfTrue) applies, so that there is an answer $a''$ such that $t_2 \Rightarrow a''$ and $a' = a''$. But $P(t_2, a)$, and thus $a = a'' = a'$.

(IfFalse) Similar to (IfTrue).

(IfNum) Suppose $t_1$, $t_2$ and $t_3$ are terms, $nv$ is a numeric value and $t_1 \Rightarrow nv$, and assume the inductive hypothesis, $P(t_1, nv)$. We must show that $P(\text{if } t_1 \text{ then } t_2 \text{ else } t_3, \text{error})$. Suppose $a$ is an answer and if $t_1$ then $t_2$ else $t_3 \Rightarrow a$. We must show that $\text{error} = a$. Because $P(t_1, \text{nv})$, it follows that $t_1$ doesn’t completely evaluate to a boolean value or error. Thus, applying the inversion lemma for the complete evaluation relation, only case (IfNum) applies, so that $a = \text{error}$, i.e., $\text{error} = a$.

(IfError) Suppose $t_1$, $t_2$ and $t_3$ are terms and $t_1 \Rightarrow \text{error}$, and assume the inductive hypothesis, $P(t_1, \text{error})$. We must show that $P(\text{if } t_1 \text{ then } t_2 \text{ else } t_3, \text{error})$. Suppose $a$ is an answer and if $t_1$ then $t_2$ else $t_3 \Rightarrow a$. We must show that $\text{error} = a$. Because $P(t_1, \text{error})$, it follows that $t_1$ doesn’t completely evaluate to a numeric or boolean value. Thus, applying the inversion lemma for the complete evaluation relation, only case (IfError) applies, so that $a = \text{error}$, i.e., $\text{error} = a$.

(SuccBool) Suppose $t$ is a term, $bv$ is a boolean value and $t \Rightarrow bv$, and assume the inductive hypothesis, $P(t, bv)$. We must show that $P(\text{suc}t, bv)$. Suppose $a$ is an answer and suc $t \Rightarrow a$. We must show that $\text{error} = a$. Because $P(t, bv)$, it follows that $t$ doesn’t completely evaluate to a numeric value or error. Thus, applying the inversion lemma for the complete evaluation relation, only case (SuccBool) applies, so that $a = \text{error}$, i.e., $\text{error} = a$. 

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(SuccNum) Suppose \( t \) is a term, \( nv \) is a numeric value and \( t \Rightarrow nv \), and assume the inductive hypothesis, \( P(t, nv) \). We must show that \( P(succ\ t, succ\ nv) \). Suppose \( a \) is an answer and \( succ\ t \Rightarrow a \). We must show that \( succ\ nv = a \). Because \( P(t, nv) \), it follows that \( t \) doesn’t completely evaluate to a boolean or \( error \). Thus, applying the inversion lemma for the complete evaluation relation, only case (SuccNum) applies, so there is a numeric value \( nv' \) such that \( t \Rightarrow nv' \) and \( a = succ\ nv' \). But \( P(t, nv) \), and thus \( nv = nv' \). Thus \( succ\ nv = succ\ nv' = a \).

(SuccError) Suppose \( t \) is a term and \( t \Rightarrow error \), and assume the inductive hypothesis, \( P(t, error) \). We must show that \( P(succ\ t, error) \). Suppose \( a \) is an answer and \( succ\ t \Rightarrow a \). We must show that \( error = a \). Because \( P(t, error) \), it follows that \( t \) doesn’t completely evaluate to a boolean value, or numeric value. Thus, applying the inversion lemma for the complete evaluation relation, only case (SuccError) applies, so that \( a = error \), i.e., \( error = a \).

(PredBool) Similar to (SuccBool).

(PredZero) Suppose \( t \) is a term and \( t \Rightarrow 0 \), and assume the inductive hypothesis, \( P(t, 0) \). We must show that \( P(pred\ t, error) \). Suppose \( a \) is an answer and \( pred\ t \Rightarrow a \). We must show that \( error = a \). Because \( P(t, 0) \), it follows that \( t \) doesn’t completely evaluate to a boolean value, a numeric value other than 0, or \( error \). Thus, applying the inversion lemma for the complete evaluation relation, only case (PredZero) applies, so that \( a = error \), i.e., \( error = a \).

(PredSucc) Suppose \( t \) is a term, \( nv \) is a numeric value and \( t \Rightarrow succ\ nv \), and assume the inductive hypothesis, \( P(t, succ\ nv) \). We must show that \( P(pred\ t, nv) \). Suppose \( a \) is an answer and \( pred\ t \Rightarrow a \). We must show that \( nv = a \). Because \( P(t, succ\ nv) \), it follows that \( t \) doesn’t completely evaluate to a boolean value, a numeric value other than 0, or \( error \). Thus, applying the inversion lemma for the complete evaluation relation, only case (PredSucc) applies, so that there is a numeric value \( nv' \) such that \( t \Rightarrow succ\ nv' \) and \( a = nv' \). But \( P(t, succ\ nv) \), and thus \( succ\ nv = succ\ nv' \), so that \( nv = nv' = a \).

(PredError) Similar to (SuccError).

(IszeroBool) Similar to (SuccBool).

(IszeroZero) Suppose \( t \) is a term and \( t \Rightarrow 0 \), and assume the inductive hypothesis, \( P(t, 0) \). We must show that \( P(iszero\ t, true) \). Suppose \( a \) is an answer and \( iszero\ t \Rightarrow a \). We must show that \( true = a \). Because \( P(t, 0) \), it follows that \( t \) doesn’t completely evaluate to a boolean value, a numeric value other than 0, or \( error \). Thus, applying the inversion lemma for the complete evaluation relation, only case (IszeroZero) applies, so that \( a = true \), i.e., \( true = a \).

(IszeroSucc) Suppose \( t \) is a term, \( nv \) is a numeric value and \( t \Rightarrow succ\ nv \), and assume the inductive hypothesis, \( P(t, succ\ nv) \). We must show that \( P(iszero\ t, false) \). Suppose \( a \) is an answer and \( iszero\ t \Rightarrow a \). We must show that \( false = a \). Because \( P(t, succ\ nv) \), it follows that \( t \) doesn’t completely evaluate to a boolean value, 0 or \( error \). Thus, applying the inversion lemma for the complete evaluation relation, only case (PredSucc) applies, so that \( a = false \), i.e., \( false = a \).

(IszeroError) Similar to (SuccError).
Suppose \( t_1 \) and \( t_2 \) are terms, \( v \) is a value and \( t_1 \Rightarrow v \), and assume the inductive hypothesis, \( P(t_1, v) \). We must show that \( P(t_1 \text{ otherwise } t_2, v) \). Suppose \( a \) is a value and \( t_1 \text{ otherwise } t_2 \Rightarrow a \). We must show that \( v = a \). Because \( P(t_1, v) \), it follows that \( t_1 \) doesn’t completely evaluate to \text{error}. Thus, applying the inversion lemma for the complete evaluation relation, only case (OtherwiseValue) applies, so that there is a value \( v' \) such that \( t_1 \Rightarrow v' \) and \( a = v' \). But \( P(t_1, v) \), and thus \( v = v' = a \).

(OtherwiseError) Suppose \( t_1 \) and \( t_2 \) are terms, \( a \) is an answer, \( t_1 \Rightarrow \text{error} \) and \( t_2 \Rightarrow a \), and assume the inductive hypothesis, \( P(t_1, \text{error}) \) and \( P(t_2, a) \). We must show that \( P(t_1 \text{ otherwise } t_2, a) \). Suppose \( a' \) is an answer and \( t_1 \text{ otherwise } t_2 \Rightarrow a' \). We must show that \( a = a' \). Because \( P(t_1, \text{error}) \), it follows that \( t_1 \) does not completely evaluate to a value. Thus, applying the inversion lemma for the complete evaluation relation, only case (OtherwiseError) applies, so that there is an answer \( a'' \) such that \( t_2 \Rightarrow a'' \) and \( a' = a'' \). But \( P(t_2, a) \), and thus \( a = a'' = a' \).

Exercise 3

Lemma 3.1
For all answers \( a \), \( a \Rightarrow a \).

Proof. By rules (CE-True), (CE-False) and (CE-Error), we have that \( \text{true} \Rightarrow \text{true} \), \( \text{false} \Rightarrow \text{false} \) and \( \text{error} \Rightarrow \text{error} \). Thus it remains to show that, for all numeric values \( nv \), \( nv \Rightarrow nv \). Define a predicate \( P \) on numeric values by: \( P(nv) \) iff \( nv \Rightarrow nv \). It will suffice to show that, for all numeric values \( nv \), \( P(nv) \). We proceed using the principle of structural induction on numeric values.

(Zero) We must show \( P(0) \), i.e., \( 0 \Rightarrow 0 \), and this follows by rule (CE-Zero).

(Successor) Suppose \( nv \) is a numeric value, and assume the inductive hypothesis, \( P(nv) \). We must show that \( P(\text{succ } nv) \). Because \( P(nv) \), we have that \( nv \Rightarrow nv \), so that \( \text{succ } nv \Rightarrow \text{succ } nv \), by rule (CE-SuccNum). Thus \( P(\text{succ } nv) \).

Lemma 3.2
For all terms \( t_1 \) and \( t_2 \) and answers \( a \), if \( t_1 \Rightarrow t_2 \Rightarrow a \), then \( t_1 \Rightarrow a \).

Proof. Define a predicate \( P \) on pairs of terms by: \( P(t_1, t_2) \) iff, for all answers \( a \), if \( t_2 \Rightarrow a \), then \( t_1 \Rightarrow a \). It will suffice to show that, for all terms \( t_1 \) and \( t_2 \), if \( t_1 \rightarrow t_2 \), then \( P(t_1, t_2) \). (To see that this is so, suppose \( t_1 \) and \( t_2 \) are terms, \( a \) is an answer and \( t_1 \rightarrow t_2 \Rightarrow a \). Then \( P(t_1, t_2) \). But \( t_2 \Rightarrow a \), and thus \( t_1 \Rightarrow a \).) We proceed using the principle of induction on the evaluation relation.

(IfTrue) Suppose \( t_2 \) and \( t_3 \) are terms. We must show that \( P(\text{if true then } t_2 \text{ else } t_3, t_2) \). Suppose \( a \) is an answer, and \( t_2 \Rightarrow a \). We must show that if \( \text{if true then } t_2 \text{ else } t_3 \Rightarrow a \). By rule (CE-True), we have that \( \text{true} \Rightarrow \text{true} \). Then, since \( \text{true} \Rightarrow \text{true} \) and \( t_2 \Rightarrow a \), rule (CE-IfTrue) shows us that if \( \text{if true then } t_2 \text{ else } t_3 \Rightarrow a \).

(IfFalse) Similar to (IfTrue).
We must show that \( P(\text{if } \text{nv} \text{ then } \text{t}_2 \text{ else } \text{t}_3, \text{error}) \). Suppose \( a \) is an answer, and \( \text{error} \Rightarrow a \). We must show that if \( \text{nv} \Rightarrow \text{t}_2 \text{ else } \text{t}_3 \Rightarrow a \). By rule (CE-Error), we have that \( \text{error} \Rightarrow \text{error} \). Since \( \text{error} \Rightarrow \text{error} \) and \( \text{error} \Rightarrow a \), Exercise 2 tells us that \( \text{error} = a \). By Lemma 3.1, we have that \( \text{nv} \Rightarrow \text{nv} \). Thus rule (CE-IfNum) shows us that if \( \text{nv} \Rightarrow \text{t}_2 \text{ else } \text{t}_3 \Rightarrow \text{error} = a \).

(IfNum) Suppose \( \text{nv} \) is a numeric value and \( \text{t}_2 \) and \( \text{t}_3 \) are terms. We must show that 

\[ P(\text{if } \text{nv} \text{ then } \text{t}_2 \text{ else } \text{t}_3, \text{error}) \]  

Suppose \( a \) is an answer, and \( \text{error} \Rightarrow a \). We must show that if \( \text{nv} \Rightarrow \text{t}_2 \text{ else } \text{t}_3 \Rightarrow a \). By rule (CE-Error), we have that \( \text{error} \Rightarrow \text{error} \). Since \( \text{error} \Rightarrow \text{error} \) and \( \text{error} \Rightarrow a \), Exercise 2 tells us that \( \text{error} = a \). By Lemma 3.1, we have that \( \text{nv} \Rightarrow \text{nv} \). Thus rule (CE-IfNum) shows us that if \( \text{nv} \Rightarrow \text{t}_2 \text{ else } \text{t}_3 \Rightarrow \text{error} = a \).

(IfError) Similar to (IfNum).

(If) Suppose \( \text{t}_1, \text{t}_1', \text{t}_2 \) and \( \text{t}_3 \) are terms, \( \text{t}_1 \rightarrow \text{t}_1' \), and assume the inductive hypothesis, \( P(\text{t}_1, \text{t}_1') \). We must show that \( P(\text{if } \text{t}_1 \text{ then } \text{t}_2 \text{ else } \text{t}_3, \text{if } \text{t}_1' \text{ then } \text{t}_2 \text{ else } \text{t}_3) \). Suppose \( a \) is an answer, and if \( \text{t}_1' \Rightarrow \text{t}_2 \text{ else } \text{t}_3 \Rightarrow a \). We must show that if \( \text{t}_1 \Rightarrow \text{t}_2 \text{ else } \text{t}_3 \Rightarrow a \). Applying the inversion lemma for the complete evaluation lemma to if \( \text{t}_1' \text{ then } \text{t}_2 \text{ else } \text{t}_3 \Rightarrow a \), there are four cases to consider.

(IfTrue) Suppose \( \text{t}_1' \Rightarrow \text{true} \) and \( \text{t}_2 \Rightarrow a \). Since \( P(\text{t}_1, \text{t}_1') \), we have that \( \text{t}_1 \Rightarrow \text{true} \). Thus, because \( \text{t}_1 \Rightarrow \text{true} \) and \( \text{t}_2 \Rightarrow a \), rule (CE-IfTrue) shows us that if \( \text{t}_1 \Rightarrow \text{t}_2 \text{ else } \text{t}_3 \Rightarrow a \).

(IfFalse) Similar to (IfTrue).

(IfNum) Suppose there is a numeric value \( \text{nv} \) such that \( \text{t}_1' \Rightarrow \text{nv} \) and \( a = \text{error} \). Since \( P(\text{t}_1, \text{t}_1') \), we have that \( \text{t}_1 \Rightarrow \text{nv} \). Thus rule (CE-IfNum) shows us that if \( \text{t}_1 \Rightarrow \text{t}_2 \text{ else } \text{t}_3 \Rightarrow \text{error} = a \).

(IfError) Suppose \( \text{t}_1' \Rightarrow \text{error} \) and \( a = \text{error} \). Since \( P(\text{t}_1, \text{t}_1') \), we have that \( \text{t}_1 \Rightarrow \text{error} \). Thus rule (CE-IfError) shows us that if \( \text{t}_1 \Rightarrow \text{t}_2 \text{ else } \text{t}_3 \Rightarrow \text{error} = a \).

(SuccBool) Suppose \( \text{bv} \) is a boolean value. We must show that \( P(\text{succ } \text{bv}, \text{error}) \). Suppose \( a \) is an answer, and \( \text{error} \Rightarrow a \). We must show that \( \text{succ } \text{bv} \Rightarrow a \). By rule (CE-Error) and Exercise 2, we have that \( a = \text{error} \). By Lemma 3.1, we have that \( \text{bv} \Rightarrow \text{bv} \). Thus, by rule (CE-SuccBool), we have that \( \text{succ } \text{bv} \Rightarrow \text{error} = a \).

(SuccError) We must show that \( P(\text{succ } \text{error}, \text{error}) \). Suppose \( a \) is an answer, and \( \text{error} \Rightarrow a \). We must show that \( \text{succ } \text{error} \Rightarrow a \). By rule (CE-Error) and Exercise 2, we have that \( a = \text{error} \). By rule (CE-Error), we have that \( \text{error} \Rightarrow \text{error} \). Thus, by rule (CE-SuccError), we have that \( \text{succ } \text{error} \Rightarrow \text{error} = a \).

(Succ) Suppose \( \text{t}_1 \) and \( \text{t}_1' \) are terms, \( \text{t}_1 \rightarrow \text{t}_1' \) and assume the inductive hypothesis, \( P(\text{t}_1, \text{t}_1') \). We must show that \( P(\text{succ } \text{t}_1, \text{succ } \text{t}_1') \). Suppose \( a \) is an answer, and \( \text{succ } \text{t}_1' \Rightarrow a \). We must show that \( \text{succ } \text{t}_1 \Rightarrow a \). Applying the inversion lemma for the complete evaluation relation to \( \text{succ } \text{t}_1' \Rightarrow a \), there are three cases to consider.

(SuccBool) Suppose there is a boolean value \( \text{bv} \) such that \( \text{t}_1' \Rightarrow \text{bv} \) and \( a = \text{error} \). Since \( P(\text{t}_1, \text{t}_1') \), it follows that \( \text{t}_1 \Rightarrow \text{bv} \). Thus rule (CE-SuccBool) shows us that \( \text{succ } \text{t}_1 \Rightarrow \text{error} = a \).

(SuccNum) Suppose there is a numeric value \( \text{nv} \) such that \( \text{t}_1' \Rightarrow \text{nv} \) and \( a = \text{succ } \text{nv} \). Since \( P(\text{t}_1, \text{t}_1') \), it follows that \( \text{t}_1 \Rightarrow \text{nv} \). Thus rule (CE-SuccNum) shows us that \( \text{succ } \text{t}_1 \Rightarrow \text{succ } \text{nv} = a \).

(SuccError) Suppose \( \text{t}_1' \Rightarrow \text{error} \) and \( a = \text{error} \). Since \( P(\text{t}_1, \text{t}_1') \), it follows that \( \text{t}_1 \Rightarrow \text{error} \). Thus rule (CE-SuccError) shows us that \( \text{succ } \text{t}_1 \Rightarrow \text{error} = a \).
(PredBool) Similar to (SuccBool).

(PredZero) We must show that \( P(\text{pred } 0, \text{error}) \). Suppose \( a \) is an answer, and \( \text{error} \Rightarrow a \). We must show that \( \text{pred } 0 \Rightarrow a \). By rule (CE-Error) and Exercise 2, we have that \( a = \text{error} \). By rule (CE-Zero), we have that \( 0 \Rightarrow 0 \). Thus, by rule (CE-PredZero), we have that \( \text{pred } 0 \Rightarrow \text{error} = a \).

(PredSucc) Suppose \( nv \) is a numeric value. We must show that \( P(\text{pred}(\text{succ } nv), nv) \). Suppose \( a \) is an answer, and \( nv \Rightarrow a \). We must show that \( \text{pred}(\text{succ } nv) \Rightarrow a \). By Lemma 3.1, we have that \( nv \Rightarrow nv \). Thus, by Exercise 2, it follows that \( a = nv \). By rule (CE-SuccNum), we have that \( \text{succ } nv \Rightarrow \text{succ } nv \), so that rule (CE-PredSucc) shows us that \( \text{pred}(\text{succ } nv) \Rightarrow nv = a \).

(PredError) Similar to (SuccError).

(Pred) Suppose \( t_1 \) and \( t_1' \) are terms, \( t_1 \rightarrow t_1' \) and assume the inductive hypothesis, \( P(t_1, t_1') \). We must show that \( P(\text{pred } t_1, \text{pred } t_1') \). Suppose \( a \) is an answer, and \( \text{pred } t_1' \Rightarrow a \). We must show that \( \text{pred } t_1 \Rightarrow a \). Applying the inversion lemma for the complete evaluation relation to \( \text{pred } t_1' \Rightarrow a \), there are four cases to consider.

(PredBool) Suppose there is a boolean value \( bv \) such that \( t_1' \Rightarrow bv \) and \( a = \text{error} \). Since \( P(t_1, t_1') \), it follows that \( t_1 \Rightarrow \text{error} \). Thus rule (CE-PredBool) shows us that \( \text{pred } t_1 \Rightarrow \text{error} = a \).

(PredZero) Suppose \( t_1' \Rightarrow 0 \) and \( a = \text{error} \). Since \( P(t_1, t_1') \), it follows that \( t_1 \Rightarrow 0 \). Thus rule (CE-PredZero) shows us that \( \text{pred } t_1 \Rightarrow \text{error} = a \).

(PredSucc) Suppose there is a numeric value \( nv \) such that \( t_1' \Rightarrow \text{succ } nv \) and \( a = nv \). Since \( P(t_1, t_1') \), it follows that \( t_1 \Rightarrow \text{succ } nv \). Thus rule (CE-PredSucc) shows us that \( \text{pred } t_1 \Rightarrow nv = a \).

(PredError) Suppose \( t_1' \Rightarrow \text{error} \) and \( a = \text{error} \). Since \( P(t_1, t_1') \), it follows that \( t_1 \Rightarrow \text{error} \). Thus rule (CE-PredError) shows us that \( \text{pred } t_1 \Rightarrow \text{error} = a \).

(IszeroBool) Similar to (SuccBool).

(IszeroZero) We must show that \( P(\text{iszero } 0, \text{true}) \). Suppose \( a \) is an answer, and \( \text{true} \Rightarrow a \). We must show that \( \text{iszero } 0 \Rightarrow a \). By rule (CE-True) and Exercise 2, we have that \( a = \text{true} \). Thus, by rules (CE-Zero) and (CE-IszeroZero), it follows that \( \text{iszero } 0 \Rightarrow \text{true} = a \).

(IszeroSucc) Suppose \( nv \) is a numeric value. We must show that \( P(\text{iszero}(\text{succ } nv), \text{false}) \). Suppose \( a \) is an answer and \( \text{false} \Rightarrow a \). We must show that \( \text{iszero}(\text{succ } nv) \Rightarrow a \). By rule (CE-False) and Exercise 2, we have that \( a = \text{false} \). By Lemma 3.1, we have that \( \text{succ } nv \Rightarrow \text{succ } nv \). Thus rule (CE-IszeroSucc) shows us that \( \text{iszero}(\text{succ } nv) \Rightarrow \text{false} = a \).

(IszeroError) Similar to (SuccError)

(Iszero) Similar to (Pred).

(OtherwiseValue) Suppose \( t \) is a term and \( v \) is a value. We must show that \( P(v \text{ otherwise } t, v) \). Suppose \( a \) is an answer, and \( v \Rightarrow a \). We must show that \( v \text{ otherwise } t \Rightarrow a \). By Lemma 3.1, we have that \( v \Rightarrow v \). Thus, by Exercise 2, it follows that \( a = v \). Hence rule (CE-OtherwiseValue) shows us that \( v \text{ otherwise } t \Rightarrow v = a \).
Lemma 3.4

(OtherwiseError) Suppose $t$ is a term. We must show that $P(error\ otherwise\ t, t)$. Suppose $a$ is an answer, and $t \Rightarrow a$. We must show that error otherwise $t \Rightarrow a$. By rule (CE-Error), we have that error $\Rightarrow$ error. Then, since error $\Rightarrow$ error and $t \Rightarrow a$, rule (CE-OtherwiseError) shows us that error otherwise $t \Rightarrow a$.

(Otherwise) Suppose $t_1$ and $t'_1$ are terms, $t_1 \rightarrow t'_1$, and assume the inductive hypothesis, $P(t_1, t'_1)$. We must show that $P(t_1\ otherwise\ t_2, t'_1\ otherwise\ t_2)$. Suppose $a$ is an answer, and $t'_1$ otherwise $t_2 \Rightarrow a$. We must show that $t_1$ otherwise $t_2 \Rightarrow a$. Applying the inversion lemma for the complete evaluation relation to $t'_1$ otherwise $t_2 \Rightarrow a$, there are two cases to consider.

(OtherwiseValue) Suppose there is a value $v$ such that $t'_1 \Rightarrow v$ and $a = v$. Since $P(t_1, t'_1)$, it follows that $t_1 \Rightarrow v$. Thus rule (CE-OtherwiseValue) shows us that $t_1$ otherwise $t_2 \Rightarrow v = a$.

(OtherwiseError) Suppose $t'_1 \Rightarrow error$ and $t_2 \Rightarrow a$. Since $P(t_1, t'_1)$, it follows that $t_1 \Rightarrow error$. Because $t_1 \Rightarrow error$ and $t_2 \Rightarrow a$, rule (CE-OtherwiseError) shows us that $t_1$ otherwise $t_2 \Rightarrow a$.

□

Lemma 3.3

For all terms $t_1$ and $t_2$ and answers $a$, if $t_1 \rightarrow^* t_2 \Rightarrow a$, then $t_1 \Rightarrow a$.

Proof. Define a predicate $P$ on pairs of terms by: $P(t_1, t_2)$ iff, for all answers $a$, if $t_2 \Rightarrow a$, then $t_1 \Rightarrow a$. It will suffice to show that, for all terms $t_1$ and $t_2$, if $t_1 \rightarrow^* t_2$, then $P(t_1, t_2)$. (To see that this is so, suppose $t_1$ and $t_2$ are terms, $a$ is an answer and $t_1 \rightarrow^* t_2 \Rightarrow a$. Then $P(t_1, t_2)$. But $t_2 \Rightarrow a$, and thus $t_1 \Rightarrow a$.) We proceed using the principle of induction on the reflexive-transitive closure of the evaluation relation.

(Eval) Suppose $t_1$ and $t_2$ are terms and $t_1 \rightarrow t_2$. We must show that $P(t_1, t_2)$. Suppose $a$ is an answer and $t_2 \Rightarrow a$. We must show that $t_1 \Rightarrow a$. Because $t_1 \rightarrow t_2 \Rightarrow a$, Lemma 3.2 tells us that $t_1 \Rightarrow a$.

(Refl) Suppose $t$ is a term. We must show that $P(t, t)$. Suppose $a$ is an answer and $t \Rightarrow a$. Then $t \Rightarrow a$, as required.

(Trans) Suppose $t_1$, $t_2$ and $t_3$ are terms, $t_1 \rightarrow^* t_2$ and $t_2 \rightarrow^* t_3$, and assume the inductive hypothesis, $P(t_1, t_2)$ and $P(t_2, t_3)$. We must show that $P(t_1, t_3)$. Suppose $a$ is an answer and $t_3 \Rightarrow a$. We must show that $t_1 \Rightarrow a$. Since $P(t_2, t_3)$ and $t_3 \Rightarrow a$, we have that $t_2 \Rightarrow a$. Then, since $P(t_1, t_2)$ and $t_2 \Rightarrow a$, we have that $t_1 \Rightarrow a$.

□

Lemma 3.4

(1) For all terms $t_1$, $t'_1$, $t_2$ and $t_3$, if $t_1 \rightarrow^* t'_1$, then if $t_1$ then $t_2$ else $t_3 \rightarrow^* if$ $t'_1$ then $t_2$ else $t_3$.

(2) For all terms $t_1$ and $t'_1$, if $t_1 \rightarrow^* t'_1$, then succ $t_1 \rightarrow^* succ t'_1$.

(3) For all terms $t_1$ and $t'_1$, if $t_1 \rightarrow^* t'_1$, then pred $t_1 \rightarrow^* pred t'_1$.
(4) For all terms \( t_1 \) and \( t'_1 \), if \( t_1 \rightarrow^* t'_1 \), then iszero \( t_1 \rightarrow^* \text{iszero} t'_1 \).

(5) For all terms \( t_1, t'_1 \) and \( t_2 \), if \( t_1 \rightarrow^* t'_1 \), then \( t_1 \) otherwise \( t_2 \rightarrow^* t'_1 \) otherwise \( t_2 \).

**Proof.** We prove Part (1), the other parts being similar.

Let \( t_2 \) and \( t_3 \) be terms. We must show that, for all terms \( t_1 \) and \( t'_1 \), if \( t_1 \rightarrow^* t'_1 \), then if \( t_1 \) then \( t_2 \) else \( t_3 \rightarrow^* \) if \( t'_1 \) then \( t_2 \) else \( t_3 \). Define a predicate \( P \) on pairs of terms by: \( P(t_1, t'_1) \) iff if \( t_1 \) then \( t_2 \) else \( t_3 \rightarrow^* \) if \( t'_1 \) then \( t_2 \) else \( t_3 \). Thus it will suffice to show that, for all terms \( t_1 \) and \( t'_1 \), if \( t_1 \rightarrow^* t'_1 \), then \( P(t_1, t'_1) \). We proceed using the principle of induction on the reflexive-transitive closure of the evaluation relation.

**Eval** Suppose \( t_1 \) and \( t'_1 \) are terms and \( t_1 \rightarrow t'_1 \). We must show that \( P(t_1, t'_1) \), i.e., if \( t_1 \) then \( t_2 \) else \( t_3 \rightarrow^* \) if \( t'_1 \) then \( t_2 \) else \( t_3 \). By rule (E-If), we have that if \( t_1 \) then \( t_2 \) else \( t_3 \rightarrow^* \) if \( t'_1 \) then \( t_2 \) else \( t_3 \). Thus if \( t_1 \) then \( t_2 \) else \( t_3 \rightarrow^* \) if \( t'_1 \) then \( t_2 \) else \( t_3 \) follows by rule (RTCE-Eval).

**Ref** Suppose \( t_1 \) is a term. We must show that \( P(t_1, t_1) \), i.e., if \( t_1 \) then \( t_2 \) else \( t_3 \rightarrow^* \) if \( t_1 \) then \( t_2 \) else \( t_3 \). And this follows by rule (RTCE-Ref).

**Trans** Suppose \( t_1, t'_1 \) and \( t''_1 \) are terms, \( t_1 \rightarrow^* t'_1 \) and \( t'_1 \rightarrow^* t''_1 \), and assume the inductive hypothesis, \( P(t_1, t'_1) \) and \( P(t'_1, t''_1) \). We must show that \( P(t_1, t''_1) \). Thus we have that if \( t_1 \) then \( t_2 \) else \( t_3 \rightarrow^* \) if \( t'_1 \) then \( t_2 \) else \( t_3 \) and if \( t'_1 \) then \( t_2 \) else \( t_3 \rightarrow^* \) if \( t''_1 \) then \( t_2 \) else \( t_3 \), so that, by rule (RTCE-Trans), it follows that if \( t_1 \) then \( t_2 \) else \( t_3 \rightarrow^* \) if \( t''_1 \) then \( t_2 \) else \( t_3 \), i.e., \( P(t_1, t''_1) \).

\[ \square \]

**Lemma 3.5**

For all terms \( t \) and answers \( a \), if \( t \Rightarrow a \), then \( t \rightarrow^* a \).

**Proof.** Define a predicate \( P \) between a term and an answer: \( P(t, a) \) iff \( t \rightarrow^* a \). It will suffice to show that, for all terms \( t \) and answers \( a \), if \( t \Rightarrow a \), then \( P(t, a) \). We proceed using the principle of induction on the complete evaluation relation.

**True** We must show that \( P(\text{true, true}) \), i.e., \( \text{true} \rightarrow^* \text{true} \), and this follows by rule (RTCE-Ref).

**False** Similar to (True).

**Zero** Similar to (True).

**Error** Similar to (True).

**IfTrue** Suppose \( t_1, t_2 \) and \( t_3 \) are terms, \( a \) is an answer, \( t_1 \Rightarrow \text{true} \) and \( t_2 \Rightarrow a \), and assume the inductive hypothesis, \( P(t_1, \text{true}) \) and \( P(t_2, a) \). We must show that \( P(\text{if } t_1 \text{ then } t_2 \text{ else } t_3, a) \). Because \( P(t_1, \text{true}) \) and \( P(t_2, a) \), we have that \( t_1 \rightarrow^* \text{true} \) and \( t_2 \rightarrow^* a \). Thus, by Lemma 3.4(1) and rule (E-IfTrue), we have that

\[
\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \rightarrow^* \text{if true then } t_2 \text{ else } t_3 \rightarrow^* a.
\]

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that if \( t_1 \) then \( t_2 \) else \( t_3 \rightarrow^* a \), i.e., \( P(\text{if } t_1 \text{ then } t_2 \text{ else } t_3, a) \).
(IfFalse) Similar to (IfTrue), but using (E-IfFalse).

(IfNum) Suppose \( t_1, t_2 \) and \( t_3 \) are terms, \( nv \) is a numeric value and \( t_1 \Rightarrow nv \), and assume the inductive hypothesis, \( P(t_1, nv) \). We must show that \( P(if \ t_1 \ then \ t_2 \ else \ t_3, error) \). Because \( P(t_1, nv) \), we have that \( t_1 \rightarrow^* nv \). Thus, by Lemma 3.4(1) and rule (E-IfNum), we have that

\[
\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \rightarrow^* \text{ if } nv \text{ then } t_2 \text{ else } t_3 \rightarrow \text{ error}.
\]

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that \( if \ t_1 \ then \ t_2 \ else \ t_3 \rightarrow^* \text{ error} \), i.e., \( P(if \ t_1 \ then \ t_2 \ else \ t_3, error) \).

(IfError) Suppose \( t_1, t_2 \) and \( t_3 \) are terms and \( t_1 \Rightarrow error \), and assume the inductive hypothesis, \( P(t_1, error) \). We must show that \( P(if \ t_1 \ then \ t_2 \ else \ t_3, error) \). Because \( P(t_1, error) \), we have that \( t_1 \rightarrow^* error \). Thus, by Lemma 3.4(1) and rule (E-IfError), we have that

\[
\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \rightarrow^* \text{ if error then } t_2 \text{ else } t_3 \rightarrow \text{ error}.
\]

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that \( if \ t_1 \ then \ t_2 \ else \ t_3 \rightarrow^* \text{ error} \), i.e., \( P(if \ t_1 \ then \ t_2 \ else \ t_3, error) \).

(SuccBool) Suppose \( t \) is a term, \( bv \) is a boolean value and \( t \Rightarrow bv \), and assume the inductive hypothesis, \( P(t, bv) \). We must show that \( P(succ \ t, error) \). Because \( P(t, bv) \), we have that \( t \rightarrow^* bv \). Thus, by Lemma 3.4(2) and rule (E-SuccBool), we have that

\[
succ \ t \rightarrow^* succ \ bv \rightarrow \text{ error}.
\]

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that \( succ \ t \rightarrow^* \text{ error} \), i.e., \( P(succ \ t, error) \).

(SuccNum) Suppose \( t \) is a term, \( nv \) is a numeric value and \( t \Rightarrow nv \), and assume the inductive hypothesis, \( P(t, nv) \). We must show that \( P(succ \ t, succ \ nv) \). Because \( P(t, nv) \), we have that \( t \rightarrow^* nv \). Thus, by Lemma 3.4(2), we have that \( succ \ t \rightarrow^* succ \ nv \), i.e., \( P(succ \ t, succ \ nv) \).

(SuccError) Suppose \( t \) is a term and \( t \Rightarrow error \), and assume the inductive hypothesis, \( P(t, error) \). We must show that \( P(succ \ t, error) \). Because \( P(t, error) \), we have that \( t \rightarrow^* error \). Thus, by Lemma 3.4(2) and rule (E-SuccError), we have that

\[
succ \ t \rightarrow^* succ \ error \rightarrow \text{ error}.
\]

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that \( succ \ t \rightarrow^* \text{ error} \), i.e., \( P(succ \ t, error) \).

(PredBool) Similar to (SuccBool).

(PredZero) Suppose \( t \) is a term and \( t \Rightarrow 0 \), and assume the inductive hypothesis, \( P(t, 0) \). We must show that \( P(ped \ t, error) \). Because \( P(t, 0) \), we have that \( t \rightarrow^* 0 \). Thus, by Lemma 3.4(3) and rule (E-PredZero), we have that

\[
pred \ t \rightarrow^* pred \ 0 \rightarrow \text{ error}.
\]

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that \( pred \ t \rightarrow^* \text{ error} \), i.e., \( P(pred \ t, error) \).
(PredSucc) Suppose $t$ is a term, $nv$ is a numeric value and $t \Rightarrow succ\ nv$, and assume the inductive hypothesis, $P(t, succ\ nv)$. We must show that $P(pred\ t, nv)$. Because $P(t, succ\ nv)$, we have that $t \rightarrow^* succ\ nv$. Thus, by Lemma 3.4(3) and rule (E-PredSucc), we have that

$$pred\ t \rightarrow^* pred(succ\ nv) \rightarrow nv.$$ 

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that $pred\ t \rightarrow^* nv$, i.e., $P(pred\ t, nv)$.

(PredError) Similar to (SuccError).

(IszeroBool) Similar to (SuccBool).

(IszeroZero) Similar to (PredZero).

(IszeroSucc) Similar to (PredSucc).

(IszeroError) Similar to (SuccError).

(OtherwiseValue) Suppose $t_1$ and $t_2$ are terms, $v$ is a value and $t_1 \Rightarrow v$, and assume the inductive hypothesis, $P(t_1, v)$. We must show that $P(t_1 otherwise t_2, v)$. Because $P(t_1, v)$, we have that $t_1 \rightarrow^* v$. Thus, by Lemma 3.4(5) and rule (E-OtherwiseValue), we have that

$$t_1 otherwise t_2 \rightarrow^* v otherwise t_2 \rightarrow v.$$ 

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that $t_1 otherwise t_2 \rightarrow^* v$, i.e., $P(t_1 otherwise t_2, v)$.

(OtherwiseError) Suppose $t_1$ and $t_2$ are terms, $a$ is an answer, $t_1 \Rightarrow error$ and $t_2 \Rightarrow a$, and assume the inductive hypothesis, $P(t_1, error)$ and $P(t_2, a)$. We must show that $P(t_1 otherwise t_2, a)$. Because $P(t_1, error)$ and $P(t_2, a)$, we have that $t_1 \rightarrow^* error$ and $t_2 \rightarrow^* a$. Thus, by Lemma 3.4(5) and rule (E-OtherwiseError), we have that

$$t_1 otherwise t_2 \rightarrow^* error otherwise t_2 \rightarrow t_2 \rightarrow^* a.$$ 

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that $t_1 otherwise t_2 \rightarrow^* a$, i.e., $P(t_1 otherwise t_2, a)$.

□

To prove that $\sim = \Rightarrow$, it will suffice to show that, for all terms $t$ and answers $a$, $t \sim a$ iff $t \Rightarrow a$. Suppose $t$ is a term and $a$ is an answer.

- Suppose $t \sim a$. Then $t \rightarrow^* a$. By Lemma 3.1, we have that $a \Rightarrow a$. Thus $t \rightarrow^* a \Rightarrow a$, so that $t \Rightarrow a$, by Lemma 3.3.

- Suppose $t \Rightarrow a$. Lemma 3.5 shows us that $t \rightarrow^* a$, so that $t \sim a$. 

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