Exercise 1

Lemma 1.1
For all contexts $\Gamma$, $\lambda_E$ terms $t$ and types $T$, if $\Gamma \vdash t : T$, then $\Gamma \vdash e(t) : T$.

Proof. Let $P(\Gamma,t,T)$ be the predicate between contexts, $\lambda_E$ terms and types that is defined by: $P(\Gamma,t,T) \iff \Gamma \vdash e(t) : T$. We use induction on the typing relation of the external language to show that, for all contexts $\Gamma$, $\lambda_E$ terms $t$ and types $T$, if $\Gamma \vdash t : T$, then $P(\Gamma,t,T)$.

(T-True) Suppose $\Gamma$ is a context. We must show that $P(\Gamma,\text{true},\text{Bool})$. We have that $\Gamma \vdash \text{true} : \text{Bool}$. And $e(\text{true}) = \text{true}$, so that $\Gamma \vdash e(\text{true}) : \text{Bool}$, i.e., $P(\Gamma,\text{true},\text{Bool})$.

(T-False) Suppose $\Gamma$ is a context. We must show that $P(\Gamma,\text{false},\text{Bool})$. We have that $\Gamma \vdash \text{false} : \text{Bool}$. And $e(\text{false}) = \text{false}$, so that $\Gamma \vdash e(\text{false}) : \text{Bool}$, i.e., $P(\Gamma,\text{false},\text{Bool})$.

(T-If) Suppose $\Gamma$ is a context, $t_1$, $t_2$ and $t_3$ are $\lambda_E$ terms, $T$ is a type, $\Gamma \vdash t_1 : \text{Bool}$, $\Gamma \vdash t_2 : T$ and $\Gamma \vdash t_3 : T$, and assume the inductive hypothesis, $P(\Gamma,t_1,\text{Bool})$, $P(\Gamma,t_2,T)$ and $P(\Gamma,t_3,T)$. We must show that $P(\Gamma,\text{if } t_1 \text{ then } t_2 \text{ else } t_3,T)$. By the inductive hypothesis, we have that $\Gamma \vdash e(t_1) : \text{Bool}$, $\Gamma \vdash e(t_2) : T$ and $\Gamma \vdash e(t_3) : T$. Thus $\Gamma \vdash e(\text{if } t_1 \text{ then } t_2 \text{ else } t_3) : T$. But $e(\text{if } t_1 \text{ then } t_2 \text{ else } t_3) = e(t_1)$ if $e(t_1)$ else $e(t_2)$ else $e(t_3)$, and thus $\Gamma \vdash e(\text{if } t_1 \text{ then } t_2 \text{ else } t_3) : T$, i.e., $P(\Gamma,\text{if } t_1 \text{ then } t_2 \text{ else } t_3,T)$.

(T-Var) Suppose $\Gamma$ is a context, $x$ is a variable, $T$ is a type, and $(x,T) \in \Gamma$. We must show that $P(\Gamma,x,T)$. Since $(x,T) \in \Gamma$, we have that $\Gamma \vdash x : T$. But $e(x) = x$, and thus $\Gamma \vdash e(x) : T$, i.e., $P(\Gamma,x,T)$.

(T-Abs) Suppose $\Gamma$ is a context, $x$ is a variable, $T_1$ and $T_2$ are types, $t$ is a $\lambda_E$ term, and $\Gamma[x \rightarrow T_1] \vdash t : T_2$, and assume the inductive hypothesis, $P(\Gamma[x \rightarrow T_1],t,T_2)$. We must show that $P(\Gamma,\lambda x : T_1.t,T_1 \rightarrow T_2)$. By the inductive hypothesis, we have that $\Gamma[x \rightarrow T_1] \vdash e(t) : T_2$, so that $\Gamma \vdash \lambda x : T_1.e(t) : T_1 \rightarrow T_2$. But $e(\lambda x : T_1.t) = \lambda x : T_1.e(t)$, and thus $\Gamma \vdash e(\lambda x : T_1.t) : T_1 \rightarrow T_2$, i.e., $P(\Gamma,\lambda x : T_1.t,T_1 \rightarrow T_2)$.

(T-WildcardAbs) Suppose $\Gamma$ is a context, $T_1$ and $T_2$ are types, $t$ is a $\lambda_E$ term, and $\Gamma \vdash t : T_2$, and assume the inductive hypothesis, $P(\Gamma,t,T_2)$. We must show that $P(\Gamma,\lambda - : T_1.t,T_1 \rightarrow T_2)$. By the inductive hypothesis, we have that $\Gamma \vdash e(t) : T_2$, so that $\Gamma \vdash \lambda - : T_1.e(t) : T_1 \rightarrow T_2$. But $e(\lambda - : T_1.t) = \lambda - : T_1.e(t)$, and thus $\Gamma \vdash e(\lambda - : T_1.t) : T_1 \rightarrow T_2$, i.e., $P(\Gamma,\lambda - : T_1.t,T_1 \rightarrow T_2)$.

(T-App) Suppose $\Gamma$ is a context, $t_1$ and $t_2$ are $\lambda_E$ terms, $T_1$ and $T_2$ are types, $\Gamma \vdash t_1 : T_1 \rightarrow T_2$ and $\Gamma \vdash t_2 : T_1$, and assume the inductive hypothesis, $P(\Gamma,t_1,T_1 \rightarrow T_2)$ and $P(\Gamma,t_2,T_1)$. We must show that $P(\Gamma,t_1 \rightarrow t_2,T_2)$. By the inductive hypothesis, we have that $\Gamma \vdash e(t_1) : T_1 \rightarrow T_2$ and $\Gamma \vdash e(t_2) : T_1$, so that $\Gamma \vdash e(t_1) e(t_2) : T_2$. But $e(t_1 \rightarrow t_2) = e(t_1) e(t_2)$, and thus $\Gamma \vdash e(t_1 \rightarrow t_2) : T_2$, i.e., $P(\Gamma,t_1 \rightarrow t_2, T_2)$.
(T-Ascribe) Suppose $\Gamma$ is a context, $t$ is a $\lambda_E$ term, $T$ is a type, and $\Gamma \vdash_E t : T$, and assume the inductive hypothesis, $P(\Gamma, t, T)$. We must show that $P(\Gamma, t as T, T)$, i.e., $\Gamma \vdash_I e(t as T) : T$. Since $e(t as T) = (\lambda x : \text{Bool} \rightarrow T . x \text{true})(\lambda \neg : \text{Bool} . e(t))$, where $x$ is the least variable, we must show that $\Gamma \vdash_I (\lambda x : \text{Bool} \rightarrow T . x \text{true})(\lambda \neg : \text{Bool} . e(t)) : T$. Since $(x, \text{Bool} \rightarrow T) \in \Gamma[x \mapsto \text{Bool} \rightarrow T]$, we have that $\Gamma[x \mapsto \text{Bool} \rightarrow T] \vdash_I x : \text{Bool} \rightarrow T$. And $\Gamma[x \mapsto \text{Bool} \rightarrow T] \vdash_I \text{true} : \text{Bool}$, so that $\Gamma[x \mapsto \text{Bool} \rightarrow T] \vdash_I x \text{true} : T$. Hence $\Gamma \vdash_I \lambda x : \text{Bool} \rightarrow T . x \text{true} : (\text{Bool} \rightarrow T) \rightarrow T$. By the induction hypothesis, we have that $\Gamma \vdash_I e(t) : T$. Thus $\Gamma \vdash_I \lambda \neg : \text{Bool} . e(t) : \text{Bool} \rightarrow T$. Finally, since $\Gamma \vdash_I \lambda x : \text{Bool} \rightarrow T . x \text{true} : (\text{Bool} \rightarrow T) \rightarrow T$ and $\Gamma \vdash_I \lambda \neg : \text{Bool} . e(t) : \text{Bool} \rightarrow T$, it follows that $\Gamma \vdash_I (\lambda x : \text{Bool} \rightarrow T . x \text{true})(\lambda \neg : \text{Bool} . e(t)) : T$.

Lemma 1.2
For all contexts $\Gamma$, $\lambda_E$ terms $t$, and types $T$, if $\Gamma \vdash_I e(t) : T$, then $\Gamma \vdash_E t : T$.

Proof. Let $P(t)$ be the predicate on $\lambda_E$ terms defined by: $P(t)$ iff, for all contexts $\Gamma$ and types $T$, if $\Gamma \vdash_I e(t) : T$, then $\Gamma \vdash_E t : T$. We use structural induction on $\lambda_E$ terms to show that, for all $\lambda_E$ terms $t$, $P(t)$.

(Constant True) We must show that $P(\text{true})$. Suppose $\Gamma$ is a context, $T$ is a type, and $\Gamma \vdash_I e(\text{true}) : T$. We must show that $\Gamma \vdash_E \text{true} : T$. Since $e(\text{true}) = \text{true}$, we have that $\Gamma \vdash_I \text{true} : T$. Thus, by the inversion lemma for the typing relation of $\lambda_I$, we have that $T = \text{Bool}$. Hence $\Gamma \vdash_E \text{true} : \text{Bool} = T$.

(Constant False) We must show that $P(\text{false})$. Suppose $\Gamma$ is a context, $T$ is a type, and $\Gamma \vdash_I e(\text{false}) : T$. We must show that $\Gamma \vdash_E \text{false} : T$. Since $e(\text{false}) = \text{false}$, we have that $\Gamma \vdash_I \text{false} : T$. Thus, by the inversion lemma for the typing relation of $\lambda_I$, we have that $T = \text{Bool}$. Hence $\Gamma \vdash_E \text{false} : \text{Bool} = T$.

(Conditional) Suppose $t_1$, $t_2$ and $t_3$ are $\lambda_E$ terms, and assume the inductive hypothesis, $P(t_1)$, $P(t_2)$ and $P(t_3)$. We must show that $P(\text{if } t_1 \text{ then } t_2 \text{ else } t_3)$. Suppose $\Gamma$ is a context, $T$ is a type, and $\Gamma \vdash_I e(\text{if } t_1 \text{ then } t_2 \text{ else } t_3) : T$. We must show that $\Gamma \vdash_E$ if $t_1$ then $t_2$ else $t_3$ : $T$. We must show that $\Gamma \vdash_I$ if $t_1$ then $t_2$ else $t_3$ : $T$. Since $e(\text{if } t_1 \text{ then } t_2 \text{ else } t_3) = e(t_1)$ then $e(t_2)$ else $e(t_3)$, we have that $\Gamma \vdash_I$ if $e(t_1)$ then $e(t_2)$ else $e(t_3)$ : $T$. By the inversion lemma for the typing relation of $\lambda_I$, we have that $\Gamma \vdash_I e(t_1) : \text{Bool}$, $\Gamma \vdash_I e(t_2) : T$ and $\Gamma \vdash_I e(t_3) : T$. Thus, by the induction hypothesis, it follows that $\Gamma \vdash_E t_1 : \text{Bool}$, $\Gamma \vdash_E t_2 : T$ and $\Gamma \vdash_E t_3 : T$. Hence $\Gamma \vdash_E$ if $t_1$ then $t_2$ else $t_3$ : $T$.

(Variable) Suppose $x$ is a variable. We must show that $P(x)$. Suppose $\Gamma$ is a context, $T$ is a type, and $\Gamma \vdash_I e(x) : T$. We must show that $\Gamma \vdash_E x : T$. Since $e(x) = x$, we have that $\Gamma \vdash_I x : T$. By the inversion lemma for the typing relation of $\lambda_I$, we have that $(x, T) \in \Gamma$. Thus $\Gamma \vdash_E x : T$.

(Abstraction) Suppose $x$ is a variable, $T$ is a type, and $t$ is a $\lambda_E$ term, and suppose the inductive hypothesis, $P(t)$. We must show that $P(\lambda x : T. t)$. Suppose $\Gamma$ is a context, $U$ is a type, and $\Gamma \vdash_I e(\lambda x : T. t) : U$. We must show that $\Gamma \vdash_E \lambda x : T. t : U$. Since $e(\lambda x : T. t) = \lambda x : T. e(t)$, we have that $\Gamma \vdash_I \lambda x : T. e(t) : U$. By the inversion lemma for the typing relation of $\lambda_I$, it
follows that there is a type $U'$ such that $U = T \rightarrow U'$ and $\Gamma[x \mapsto T] \vdash_I e(t) : U'$. Thus, by the inductive hypothesis, it follows that $\Gamma[x \mapsto T] \vdash_E t : U'$, so that $\Gamma \vdash_E \lambda x : T. t : T \rightarrow U' = U$.

**Wildcard Abstraction** Suppose $T$ is a type and $t$ is a $\lambda_E$ term, and assume the inductive hypothesis, $P(t)$. We must show that $P(t_1 t_2)$. Suppose $\Gamma$ is a context, $T$ is a type, and $\Gamma \vdash_I e(t_1 t_2) : T$. We must show that $\Gamma \vdash_E t_1 t_2 : T$. By the inversion lemma for the typing relation of $\lambda_I$, we have that there is a type $T'$ such that $\Gamma \vdash_I e(t_1) : T'$ and $\Gamma \vdash_I e(t_2) : T'$. Thus, by the inductive hypothesis, it follows that $\Gamma \vdash_E t_1 : T'$ and $\Gamma \vdash_E t_2 : T'$, so that $\Gamma \vdash_E t_1 t_2 : T$.

**Application** Suppose $t_1$ and $t_2$ are $\lambda_E$ terms, and assume the inductive hypothesis, $P(t_1)$ and $P(t_2)$. We must show that $P(t_1 t_2)$. Suppose $\Gamma$ is a context, $T$ is a type, and $\Gamma \vdash_I e(t_1 t_2) : T$. We must show that $\Gamma \vdash_E t_1 t_2 : T$. Since $e(t_1 t_2) = e(t_1) e(t_2)$, we have that $\Gamma \vdash_I e(t_1) e(t_2) : T$. By the inversion lemma for the typing relation of $\lambda_I$, we have that there is a type $T'$ such that $\Gamma \vdash_I e(t_1) : T'$ and $\Gamma \vdash_I e(t_2) : T'$. Thus, by the inductive hypothesis, it follows that $\Gamma \vdash_E t_1 : T'$ and $\Gamma \vdash_E t_2 : T'$, so that $\Gamma \vdash_E t_1 t_2 : T$.

**Ascription** Suppose $t$ is a $\lambda_E$ term and $T$ is a type, and assume the inductive hypothesis, $P(t)$. We must show that $P(t$ as $T$). Suppose $\Gamma$ is a context, $U$ is a type, and $\Gamma \vdash_I e(t$ as $T$) : $U$. We must show that $\Gamma \vdash_E t$ as $T$ : $U$. Since $e(t$ as $T) = (\lambda x : T.x \text{true})((\lambda x : \text{Boolean}. e(t)) : U$. By the inversion lemma for the typing relation of $\lambda_I$, we have that there is a type $U'$ such that $\Gamma \vdash_I \lambda x : \text{Boolean}. T. x \text{true} : U' \rightarrow U$ and $\Gamma \vdash_I \lambda x : \text{Boolean}. e(t) : U'$. Since $(x, \text{Boolean} \rightarrow T) \in \Gamma[x \mapsto \text{Boolean} \rightarrow T]$, we have that $\Gamma[x \mapsto \text{Boolean} \rightarrow T] \vdash_I x : \text{Boolean} \rightarrow T$. And $\Gamma[x \mapsto \text{Boolean} \rightarrow T] \vdash_I \text{true} : \text{Boolean}$, so that $\Gamma[x \mapsto \text{Boolean} \rightarrow T] \vdash_I x \text{true} : T$. Hence $\Gamma \vdash_I \lambda x : \text{Boolean} \rightarrow T \text{true} : (\text{Boolean} \rightarrow T) \rightarrow T$. Thus, by the Uniqueness of Typing Proposition for $\lambda_I$, it follows that $(\text{Boolean} \rightarrow T) \rightarrow T = U' \rightarrow U$, so that $U' = \text{Boolean} \rightarrow T$ and $U = T$. Thus it will suffice to show that $\Gamma \vdash_E t$ as $T$ : $T$. Since $\Gamma \vdash_I \lambda x : \text{Boolean}. e(t) : U' = \text{Boolean} \rightarrow T$, the inversion lemma for the typing relation on $\lambda_I$ tells us that $\Gamma \vdash_I e(t) : T$. Thus, by the inductive hypothesis, we have that $\Gamma \vdash_E t : T$, so that $\Gamma \vdash_E t$ as $T$ : $T$.

$\square$

Suppose $t$ is a $\lambda_E$ term. We must show that

$$\emptyset \vdash_E t : \text{Boolean} \iff \emptyset \vdash_I e(t) : \text{Boolean}.$$ 

The are two directions to show.

- Suppose $\emptyset \vdash_E t : \text{Boolean}$. By Lemma 1.1, we have that $\emptyset \vdash_I e(t) : \text{Boolean}$.
- Suppose $\emptyset \vdash_I e(t) : \text{Boolean}$. By Lemma 1.2, we have that $\emptyset \vdash_E t : \text{Boolean}$.

**Exercise 2**

**Lemma 2.1**

- **(1)** For all closed $\lambda_I$ terms $t_1, t_1', t_2$ and $t_3$, if $t_1 \rightarrow_I t_1'$, then if $t_1 \text{ then } t_2 \text{ else } t_3 \rightarrow_I t_1'$ if $t_1'$ then $t_2$ else $t_3$. 

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Lemma 2.2
For all closed \( \lambda_E \) terms \( t \) and \( t' \), if \( t \to_E t' \), then \( e(t) \to_I^* e(t') \).

Proof. Simple inductions on \( \to_I^* \). □

(2) For all closed \( \lambda_I \) terms \( t_1, t'_1 \) and \( t_2, t'_2 \), if \( t_1 \to_I^* t'_1, t_2 \to_I^* t'_2 \), then \( t_1 t_2 \to_I^* t'_1 t'_2 \).

(3) For all closed \( \lambda_I \) values \( v_1 \), and closed \( \lambda_I \) terms \( t_2 \) and \( t'_2 \), if \( t_2 \to_I^* t'_2 \), then \( v_1 t_2 \to_I^* v_1 t'_2 \).

(E-IfTrue) Suppose \( t_2 \) and \( t_3 \) are closed \( \lambda_E \) terms. We must show that \( P(\text{if true then } t_2 \text{ else } t_3, t_2) \). We have that \( e(\text{if true then } t_2 \text{ else } t_3) = e(\text{true}) e(t_2) e(t_3) = \text{if true then } e(t_2) \text{ else } e(t_3) = e(t_2) \to_I^* e(t_2) \).

(E-IfFalse) Suppose \( t_2 \) and \( t_3 \) are closed \( \lambda_E \) terms. We must show that \( P(\text{if false then } t_2 \text{ else } t_3, t_3) \). We have that \( e(\text{if false then } t_2 \text{ else } t_3) = e(\text{false}) e(t_2) e(t_3) = \text{if false then } e(t_2) \text{ else } e(t_3) = \text{if false then } e(t_2) \to_I^* e(t_3) \).

(E-If) Suppose \( t_1, t'_1, t_2, t_3 \) are closed \( \lambda_E \) terms, and \( t_1 \to_E t'_1 \), and assume the inductive hypothesis, \( P(t_1, t'_1) \), i.e., \( e(t_1) \to_I^* e(t'_1) \). We must show that \( P(\text{if } t_1 \text{ then } t_2 \text{ else } t_3, t_1) \to_I^* e(t_1) \to_I^* e(t'_1) \).

(E-App1) Suppose \( t_1, t'_1, t_2, t_3 \) are closed \( \lambda_E \) terms, and \( t_1 \to_E t'_1 \), and assume the inductive hypothesis, \( P(t_1, t'_1) \), i.e., \( e(t_1) \to_I^* e(t'_1) \). We must show that \( P(\text{if } t_1 \text{ then } t_2 \text{ else } t_3, t_1) \to_I^* e(t_1) \to_I^* e(t'_1) \).

(E-App2) Suppose \( v_1 \) is a closed \( \lambda_E \) value, \( t_2, t'_2 \) are closed \( \lambda_E \) terms, and \( t_2 \to_E t'_2 \), and assume the inductive hypothesis, \( P(t_2, t'_2) \), i.e., \( e(t_2) \to_I^* e(t'_2) \). We must show that \( P(v_1 t_2, v_1 t'_2) \).

(E-AppAbs) Suppose \( x \) is a variable, \( T \) is a type, \( t \) is a \( \lambda_E \) term such that \( \text{FV}(t) \subseteq \{x\} \), and \( v \) is a closed \( \lambda_E \) value. We must show that \( P((\lambda x : T.t)v, [x \to v]t) \). We have that \( \text{FV}(e(t)) \subseteq \{x\} \). Since \( v \) is a closed \( \lambda_E \) value, we have that \( e(v) \) is a closed \( \lambda_I \) value. Thus \( e((\lambda x : T.t)v) = e(\lambda x : T.t) e(v) = (\lambda x : T.t) e(v) \to_I^* [x \to v] e(\lambda x : T.t) e(v) \to_I^* [x \to v] e(v) \).

(E-AscribeEager) Suppose \( t \) is a closed \( \lambda_E \) term and \( T \) is a type. We must show that \( P(t \text{ as } T, t) \). We have that \( e(t \text{ as } T) = (\lambda x : \text{Bool} \to T. x \to \text{true}) (\lambda x : \text{Bool} e(t)), \) where \( x \) is the least variable. Thus \( e(t \text{ as } T) \to_I^* (\lambda x : \text{Bool} e(t)) \to_I^* \).
Lemma 2.3

For all \( \lambda_E \) terms \( t \), for all closed \( \lambda_I \) terms \( u \), if \( t \) is closed and \( e(t) \rightarrow_I u \), then there is a closed \( \lambda_E \) term \( t' \) such that \( u \rightarrow_I^* e(t') \) and \( t \rightarrow_E t' \).

Proof. Let \( P(t) \) be the predicate on \( \lambda_E \) terms defined by: \( P(t) \) iff, for all closed \( \lambda_I \) terms \( u \), if \( t \) is closed and \( e(t) \rightarrow_I u \), then there is a closed \( \lambda_E \) term \( t' \) such that \( u \rightarrow_I^* e(t') \) and \( t \rightarrow_E t' \). We use structural induction on \( \lambda_E \) terms to show that, for all \( \lambda_E \) terms \( t \), \( P(t) \).

(\textbf{Constant True}) We must show that \( P(\text{true}) \). Suppose \( u \) is a closed \( \lambda_I \) term, \( \text{true} \) is closed, and \( e(\text{true}) \rightarrow_I u \). We must show that there is a closed \( \lambda_E \) term \( t' \) such that \( u \rightarrow_I^* e(t') \) and \( \text{true} \rightarrow_E t' \). But \( e(\text{true}) = \text{true} \), and thus \( \text{true} \rightarrow_I u \) — contradiction. Thus there is a closed \( \lambda_E \) term \( t' \) such that \( u \rightarrow_I^* e(t') \) and \( \text{true} \rightarrow_E t' \).

(\textbf{Constant False}) We must show that \( P(\text{false}) \). Suppose \( u \) is a closed \( \lambda_I \) term, \( \text{false} \) is closed, and \( e(\text{false}) \rightarrow_I u \). We must show that there is a closed \( \lambda_E \) term \( t' \) such that \( u \rightarrow_I^* e(t') \) and \( \text{false} \rightarrow_E t' \). But \( e(\text{false}) = \text{false} \), and thus \( \text{false} \rightarrow_I u \) — contradiction. Thus there is a closed \( \lambda_E \) term \( t' \) such that \( u \rightarrow_I^* e(t') \) and \( \text{false} \rightarrow_E t' \).

(\textbf{Conditional}) Suppose \( t_1 \), \( t_2 \) and \( t_3 \) are \( \lambda_E \) terms, and assume the inductive hypothesis, \( P(t_1) \), \( P(t_2) \) and \( P(t_3) \). We must show that \( P(\text{if } t_1 \text{ then } t_2 \text{ else } t_3) \). Suppose \( u \) is a closed \( \lambda_I \) term, if \( t_1 \) then \( t_2 \) else \( t_3 \) is closed, and \( e(\text{if } t_1 \text{ then } t_2 \text{ else } t_3) \rightarrow_I u \). We must show that there is a closed \( \lambda_E \) term \( t' \) such that \( u \rightarrow_I^* e(t') \) and if \( t_1 \) then \( t_2 \) else \( t_3 \rightarrow_E t' \). Since if \( t_1 \) then \( t_2 \) else \( t_3 \) is closed, we have that \( t_1 \), \( t_2 \) and \( t_3 \) are closed, so that \( e(t_1) \), \( e(t_2) \) and \( e(t_3) \) are also closed. Since \( e(\text{if } t_1 \text{ then } t_2 \text{ else } t_3) = e(t_1) \text{ then } e(t_2) \text{ else } e(t_3) \), we have that if \( e(t_1) \) then \( e(t_2) \) else \( e(t_3) \rightarrow_I u \). By the inversion lemma for the evaluation relation of \( \lambda_I \), there are three cases to consider.

(\textbf{E-IfTrue}) Suppose \( e(t_1) = \text{true} \) and \( u = e(t_2) \). By the definition of the elaboration function, we have that \( t_1 \rightarrow_I \text{true} \) and if \( t_1 \) then \( t_2 \) else \( t_3 \rightarrow_E t_2 \), showing that if \( t_1 \) then \( t_2 \) else \( t_3 \rightarrow_E t_2 \).

(\textbf{E-IfFalse}) Suppose \( e(t_1) = \text{false} \) and \( u = e(t_3) \). By the definition of the elaboration function, we have that \( t_1 \rightarrow_I \text{false} \). Thus \( u \rightarrow_I^* e(t_3) \) and if \( t_1 \) then \( t_2 \) else \( t_3 \rightarrow_E t_3 \), showing that if \( t_1 \) then \( t_2 \) else \( t_3 \rightarrow_E t_3 \).

(\textbf{E-If}) Suppose there is a closed \( \lambda_I \) term \( u' \) such that \( e(t_1) \rightarrow_I u' \) and \( u = e(t_2) \text{ else } e(t_3) \). By \( P(t_1) \), we have that there is a closed \( \lambda_E \) term \( t_1' \) such that \( u' \rightarrow_I^* e(t_1') \) and \( t_1 \rightarrow_E t_1' \). Thus \( u = e(t_1) \text{ then } e(t_2) \text{ else } e(t_3) \rightarrow_I^* \) if \( e(t_1') \) then \( e(t_2) \text{ else } e(t_3) = e(t_1') \text{ then } t_2 \text{ else } t_3 \), by Lemma 2.1(1), showing that \( u \rightarrow_I^* e(t_1') \) and \( t_2 \text{ else } t_3 \). And if \( t_1 \) then \( t_2 \text{ else } t_3 \rightarrow_E t_1' \text{ then } t_2 \text{ else } t_3 \).

(\textbf{Variable}) Suppose \( x \) is a variable. We must show that \( P(x) \). Suppose \( u \) is a closed \( \lambda_I \) term, \( x \) is closed, and \( e(x) \rightarrow_I u \). We must show that there is a closed \( \lambda_E \) term \( t' \) such that \( u \rightarrow_I^* e(t') \) and \( x \rightarrow_E t' \). But \( x \) is not closed — contradiction. Thus there is a closed \( \lambda_E \) term \( t' \) such that \( u \rightarrow_I^* e(t') \) and \( x \rightarrow_E t' \).
(Abstraction) Suppose \( x \) is a variable, \( T \) is a type, and \( t \) is a \( \lambda_E \) term, and suppose the inductive hypothesis, \( P(t) \). We must show that \( P(\lambda x : T. \; t) \). Suppose \( u \) is a closed \( \lambda_I \) term, \( \lambda x : T. \; t \) is closed, and \( e(\lambda x : T. \; t) \rightarrow_I u \). We must show that there is a closed \( \lambda_E \) term \( t' \) such that \( u \rightarrow_t^* e(t') \) and \( \lambda x : T. \; t \rightarrow_E t' \). Since \( e(\lambda x : T. \; t) = \lambda x : T. \; e(t) \), we have that \( \lambda x : T. \; e(t) \rightarrow_I u \) -- contradiction. Thus there is a closed \( \lambda_E \) term \( t' \) such that \( u \rightarrow_t^* e(t') \) and \( \lambda x : T. \; t \rightarrow_E t' \).

(Wildcard Abstraction) Suppose \( T \) is a type and \( t \) is a \( \lambda_E \) term, and assume the inductive hypothesis, \( P(t) \). We must show that \( P(\lambda x : T. \; t) \). Suppose \( u \) is a closed \( \lambda_I \) term, \( \lambda x : T. \; t \) is closed, and \( e(\lambda x : T. \; t) \rightarrow_I u \). We must show that there is a closed \( \lambda_E \) term \( t' \) such that \( u \rightarrow_t^* e(t') \) and \( \lambda x : T. \; t \rightarrow_E t' \). Since \( e(\lambda x : T. \; t) = \lambda x : T. \; e(t) \), we have that \( \lambda x : T. \; e(t) \rightarrow_I u \) -- contradiction. Thus there is a closed \( \lambda_E \) term \( t' \) such that \( u \rightarrow_t^* e(t') \) and \( \lambda x : T. \; t \rightarrow_E t' \).

(Application) Suppose \( t_1 \) and \( t_2 \) are \( \lambda_E \) terms, and assume the inductive hypothesis, \( P(t_1) \) and \( P(t_2) \). We must show that \( P(t_1 \; t_2) \). Suppose \( u \) is a closed \( \lambda_I \) term, \( t_1 \; t_2 \) is closed, and \( e(t_1 \; t_2) \rightarrow_I u \). We must show that there is a closed \( \lambda_E \) term \( t' \) such that \( u \rightarrow_t^* e(t') \) and \( t_1 \; t_2 \rightarrow_E e(t') \).

Since \( t_1 \; t_2 \) is closed, we have that \( t_1 \) and \( t_2 \) are closed, so that \( e(t_1) \) and \( e(t_2) \) are also closed. Because \( e(t_1 \; t_2) = e(t_1) \; e(t_2) \), we have that \( e(t_1) \; e(t_2) \rightarrow_I u \). By the inversion lemma for the evaluation relation of \( \lambda_I \), there are four cases to consider.

(E-App1) Suppose there is a closed \( \lambda_I \) term \( u' \) such that \( e(t_1) \rightarrow_I u' \) and \( u = u' \; e(t_2) \). By \( P(t_1) \), we have that there is a closed \( \lambda_E \) term \( t_1' \) such that \( u' \rightarrow_t^* e(t_1') \) and \( t_1 \rightarrow t_1' \). Thus \( u = u' \; e(t_2) \rightarrow_t^* e(t_1') \; e(t_2) = e(t_1' \; t_2) \), by Lemma 2.1(2), showing that \( u \rightarrow_t^* e(t_1' \; t_2) \). And \( t_1 \; t_2 \rightarrow_E e(t_1' \; t_2) \).

(E-App2) Suppose \( e(t_1) \) is a \( \lambda_I \) value, and there is a closed \( \lambda_I \) term \( u' \) such that \( e(t_2) \rightarrow_I u' \) and \( u = e(t_1) \; u' \). By \( P(t_2) \), we have that there is a closed \( \lambda_E \) term \( t_2' \) such that \( u' \rightarrow_t^* e(t_2') \) and \( t_2 \rightarrow_E t_2' \). Thus \( u = e(t_1) \; u' \rightarrow_t^* e(t_1) \; e(t_2') = e(t_1' \; t_2') \), by Lemma 2.1(3), showing that \( u \rightarrow_t^* e(t_1' \; t_2') \). Since \( e(t_1) \) is a \( \lambda_I \) value, it follows that \( t_1 \) is a \( \lambda_E \) value. Hence \( t_1 \; t_2 \rightarrow_E e(t_1' \; t_2') \).

(E-AppAbs) Suppose \( e(t_2) \) is a \( \lambda_I \) value, and there is a variable \( x \), a type \( T \) and a \( \lambda_I \) term \( u' \) with the property that \( \text{FV}(u') \subseteq \{x\} \) such that \( e(t_1) = \lambda x : T. \; u' \) and \( u = [x \mapsto e(t_2)]u' \). By the definition of the elaboration function, it follows that \( t_1 = \lambda x : T. \; t' \) for some \( \lambda_E \) term \( t' \) such that \( e(t') = u' \), so that \( \text{FV}(t') \subseteq \{x\} \). Thus \( u = [x \mapsto e(t_2)]u' = [x \mapsto e(t_2)(e(t')) = e([x \mapsto e(t_2)](e(t'))) \), by the Elaboration of Substitution Proposition, showing that \( u \rightarrow_t^* e([x \mapsto e(t_2)](e(t'))) \). Since \( e(t_2) \) is a \( \lambda_I \) value, it follows that \( t_2 \) is a \( \lambda_E \) value. Thus \( t_1 \; t_2 = (\lambda x : T. \; t') \; t_2 \rightarrow_E e([x \mapsto e(t_2)](e(t'))) \).

(E-AppWildcardAbs) Suppose \( e(t_2) \) is a \( \lambda_I \) value, and there is a type \( T \) and a closed \( \lambda_I \) term \( u' \) such that \( e(t_1) = \lambda \lambda x : T. \; u' \) and \( u = u' \). By the definition of the elaboration function, it follows that \( t_1 = \lambda x : T. \; t' \) for some \( \lambda_E \) term \( t' \) such that \( e(t') = u' \), so that \( t' \) is also closed. Thus \( u = u' = e(t') \), showing that \( u \rightarrow_t^* e(t') \). Since \( e(t_2) \) is a \( \lambda_I \) value, it follows that \( t_2 \) is a \( \lambda_E \) value. Thus \( t_1 \; t_2 = (\lambda \lambda x : T. \; t') \; t_2 \rightarrow_E e(t') \).

(Ascription) Suppose \( t \) is a \( \lambda_E \) term and \( T \) is a type, and assume the inductive hypothesis, \( P(t) \). We must show that \( P(t \; as \; T) \). Suppose \( u \) is a closed \( \lambda_I \) term, \( t \; as \; T \) is closed and \( e(t \; as \; T) \rightarrow_I u \). We must show that there is a closed \( \lambda_E \) term \( t'' \) such that \( u \rightarrow_t^* e(t'') \) and \( t \; as \; T \rightarrow_E t'' \). Since \( t \; as \; T \) is closed, we have that \( t \) is closed. Since \( e(t \; as \; T) = (\lambda x : \text{Bool} \rightarrow T. \; x. \; \text{true})(\lambda \lambda x : \text{Bool} \; e(t)) \), where \( x \) is the least variable, we have that \( (\lambda x : \text{Bool} \rightarrow T. \; x. \; \text{true})(\lambda \lambda x : \text{Bool} \; e(t)) \rightarrow_I u \), and thus that \( u = (\lambda \lambda x : \text{Bool} \; e(t)) \text{true} \). Thus \( u \rightarrow_t e(t) \), showing that \( u \rightarrow_t^* e(t) \). And \( t \; as \; T \rightarrow_E t \).
\[\square\]

**Lemma 2.4**
For all closed \(\lambda_E\) terms \(t\) and \(t'\), if \(t \rightarrow^*_{E} t'\), then \(e(t) \rightarrow^*_I e(t')\).

**Proof.** Define a predicate \(P(t, t')\) on closed \(\lambda_E\) terms by: \(P(t, t')\) iff \(e(t) \rightarrow^*_I e(t')\). We use induction on \(\rightarrow^*_E\) to show that, for all closed \(\lambda_E\) terms \(t\) and \(t'\), if \(t \rightarrow^*_E t'\), then \(P(t, t')\).

**(Eval)** Suppose \(t\) and \(t'\) are closed \(\lambda_E\) terms and \(t \rightarrow^*_E t'\). We must show that \(P(t, t')\). By Lemma 2.2, we have that \(e(t) \rightarrow^*_I e(t')\), i.e., \(P(t, t')\).

**(Refl)** Suppose \(t\) is a closed \(\lambda_E\) term. Then \(e(t) \rightarrow^*_I e(t),\) i.e., \(P(t, t)\).

**(Trans)** Suppose \(t_1, t_2\) and \(t_3\) are closed \(\lambda_E\) terms, \(t_1 \rightarrow^*_E t_2\) and \(t_2 \rightarrow^*_E t_3\), and assume the inductive hypothesis, \(P(t_1, t_2)\) and \(P(t_2, t_3)\). We must show that \(P(t_1, t_3)\). By the inductive hypothesis, we have that \(e(t_1) \rightarrow^*_I e(t_2) \rightarrow^*_I e(t_3)\), i.e., \(P(t_1, t_3)\).

\[\square\]

**Lemma 2.5**
For all closed \(\lambda_I\) terms \(u\) and \(u'\), if \(u \rightarrow^*_I u'\), then for all closed \(\lambda_E\) terms \(t\), if \(u \rightarrow^*_I e(t)\), then there is a closed \(\lambda_E\) term \(t'\) such that \(u' \rightarrow^*_I e(t')\) and \(t \rightarrow^*_E t'\).

**Proof.** Define a predicate \(P(u, u')\) on closed \(\lambda_I\) terms by: \(P(u, u')\) iff, for all closed \(\lambda_E\) terms \(t\), if \(u \rightarrow^*_I e(t)\), then there is a closed \(\lambda_E\) term \(t'\) such that \(u' \rightarrow^*_I e(t')\) and \(t \rightarrow^*_E t'\). We use induction on \(\rightarrow^*_I\) to show that, for all closed \(\lambda_I\) terms \(u\) and \(u'\), if \(u \rightarrow^*_I u'\), then \(P(u, u')\).

**(Eval)** Suppose \(u\) and \(u'\) are closed \(\lambda_I\) terms and \(u \rightarrow^*_I u'\). We must show that \(P(u, u')\). Suppose \(t\) is a closed \(\lambda_E\) term and \(u \rightarrow^*_I e(t)\). We must show that there is a closed \(\lambda_E\) term \(t'\) such that \(u' \rightarrow^*_I e(t')\) and \(t \rightarrow^*_E t'\).

- Suppose \(n = 0\). Thus \(u \rightarrow^*_I e(t)\), so that \(u = e(t)\). Then \(e(t) = u \rightarrow^*_I u'\), and thus Lemma 2.3 tells us that there is a closed \(\lambda_E\) term \(t'\) such that \(u' \rightarrow^*_I e(t')\) and \(t \rightarrow^*_E t'\). Hence \(t \rightarrow^*_E t'\).

- Suppose \(n \geq 1\). Thus there is a closed \(\lambda_I\) term \(u''\) such that \(u \rightarrow^*_I u'' \rightarrow^*_I e(t)\). By the Determinacy of Evaluation Proposition for \(\lambda_I\), since \(u \rightarrow^*_I u'\) and \(u \rightarrow^*_I u''\), we have that \(u' = u''\). Thus \(u' \rightarrow^*_I e(t)\) and \(t \rightarrow^*_E t\).

**(Refl)** Suppose \(u\) is a closed \(\lambda_I\) term. We must show that \(P(u, u)\). Suppose \(t\) is a closed \(\lambda_E\) term, and \(u \rightarrow^*_I e(t)\). We must show that there is a closed \(\lambda_E\) term \(t'\) such that \(u \rightarrow^*_I e(t')\) and \(t \rightarrow^*_E t'\).

We have that \(u \rightarrow^*_I e(t)\) and \(t \rightarrow^*_E t\).

**(Trans)** Suppose \(u_1, u_2\) and \(u_3\) are closed \(\lambda_I\) terms, \(u_1 \rightarrow^*_I u_2\) and \(u_2 \rightarrow^*_I u_3\), and assume the inductive hypothesis, \(P(u_1, u_2)\) and \(P(u_2, u_3)\). We must show that \(P(u_1, u_3)\). Suppose \(t_1\) is a closed \(\lambda_E\) term, and \(u_1 \rightarrow^*_I e(t_1)\). We must show that there is a closed \(\lambda_E\) term \(t_3\) such that \(u_3 \rightarrow^*_I e(t_3)\). Since \(P(u_1, u_2)\) and \(u_1 \rightarrow^*_I e(t_1)\), there is a closed \(\lambda_E\) term \(t_2\) such that \(u_2 \rightarrow^*_I e(t_2)\) and \(t_1 \rightarrow^*_E t_2\). Since \(P(u_2, u_3)\) and \(u_2 \rightarrow^*_I e(t_2)\), there is a closed \(\lambda_E\) term \(t_3\) such that \(u_3 \rightarrow^*_I e(t_3)\) and \(t_2 \rightarrow^*_E t_3\). Since \(t_1 \rightarrow^*_E t_2 \rightarrow^*_E t_3\), we have that \(t_1 \rightarrow^*_E t_3\). Thus \(u_3 \rightarrow^*_I e(t_3)\) and \(t_1 \rightarrow^*_E t_3\).
Suppose \( t \) is a closed \( \lambda_E \) term and \( b \in \{\text{true}, \text{false}\} \). We must show that

\[
t \rightarrow^*_{\lambda} b \quad \text{iff} \quad e(t) \rightarrow^* I b.
\]

There are two directions to show.

- Suppose \( t \rightarrow^*_{\lambda} b \). By Lemma 2.4, we have that \( e(t) \rightarrow^*_I e(b) \). Since \( e(\text{true}) = \text{true} \) and \( e(\text{false}) = \text{false} \), we have that \( e(b) = b \). Thus \( e(t) \rightarrow^* I e(b) = b \).

- Suppose \( e(t) \rightarrow^* I b \). By Lemma 2.5, since \( t \) is closed and \( e(t) \rightarrow^* e(t) \), we have that there is a closed \( \lambda_E \) term \( t' \) such that \( b \rightarrow^* I e(t') \) and \( t \rightarrow^*_{\lambda} t' \). Since \( b \rightarrow^* I e(t') \), there is an \( n \in \mathbb{N} \) such that \( b \rightarrow^n I e(t') \). If \( n \geq 1 \), then \( b \) is not a normal form, which is impossible. Thus \( n = 0 \), and \( b \rightarrow^0 I e(t') \), i.e., \( b = e(t') \). By the definition of the elaboration function, we can see that, if \( b = \text{true} \), then \( t' = \text{true} \), and, if \( b = \text{false} \), then \( t' = \text{false} \). Thus \( t' = b \), so that \( t \rightarrow^*_{\lambda} t' = b \).