CIS 705 — Programming Languages — Spring 2010

Final Examination

Model Answers

Question 1

```
(* val permsOnto : 'a list * 'a list * 'a list list -> 'a list list

   if xs @ ys has no duplicates, then permsOnto(xs, ys, zss) returns
   wss @ zss, where wss is all the permutations of xs @ ys that begin
   with xs, listed in some order without duplicates

   termination: in recursive calls, the length of ys goes down *)

let rec permsOnto(xs, ys, zss) =
      match ys with
              [] -> xs :: zss
            | ys ->
                (* val choose : 'a list * 'a list * 'a list list -> 'a list list

                   if us @ vs is a permutation of ys, then choose(us, vs, zss)
                   returns wss @ zss, where wss is all the permutations of
                   xs @ ys that begin with xs @ [v], for some v in vs, listed
                   in some order without duplicates

                   termination: recursion is on vs *)

                let rec choose(us, vs, zss) =
                      match vs with
                            []       -> zss
                          | v :: vs ->
                              permsOnto(xs @ [v], us @ vs,
                                        choose(v :: us, vs, zss))
                in choose([], ys, zss)

(* val perms : 'a list -> 'a list list

   if xs has no duplicates, then perms xs returns the list of all
   permutations of xs, listed in some order without duplicates *)

let perms xs = permsOnto([], xs, [])
```

Question 2

Question A

$T = \mathsf{Nat}$ and $v = 3$.

Question B

Let $P(\Gamma, t, T)$ be the predicate between contexts, terms and types defined by: $P(\Gamma, t, T)$ iff, if $\Gamma = \emptyset$, then t is a closed term that is not stuck. We use induction on the typing relation to prove that, for all contexts Γ, terms t and types T, if $\Gamma \vdash t : T$, then $P(\Gamma, t, T)$. (Then, suppose t is a closed term, and t is well-typed. Thus $\emptyset \vdash t : T$ for some type T, so that $P(\emptyset, t, T)$. But $\emptyset = \emptyset$, and thus t is not stuck.)

(T-Nat) Suppose Γ is a context and $n \in \mathbb{N}$. We must show that $P(\Gamma, n, \mathsf{Nat})$. Suppose $\Gamma = \emptyset$. We must show that n is a closed term that is not stuck, and this follows since n is a closed value.

(T-Succ) Suppose Γ is a context, t is a term and $\Gamma \vdash t : \mathsf{Nat}$, and assume the inductive hypothesis, $P(\Gamma, t, \mathsf{Nat})$. We must show that $P(\Gamma, \mathsf{succ}\, t, \mathsf{Nat})$. Suppose $\Gamma = \emptyset$. We must show that $\mathsf{succ}\, t$ is a closed term that is not stuck. By the inductive hypothesis, we have that t is a closed term that is not stuck. Thus $\mathsf{succ}\, t$ is closed. Since t is not stuck, either t not a normal form, or t is a value, and thus there are two cases to consider.

- Suppose t is not a normal form. Then there is a closed term t' such that $t \rightarrow t'$, so that $\mathsf{succ}\, t \rightarrow \mathsf{succ}\, t'$, showing that $\mathsf{succ}\, t$ is not stuck.
- Suppose t is a value. Then, since $\emptyset \vdash t : \mathsf{Nat}$, the Canonical Forms Lemma tells us that $t = n$, for some $n \in \mathbb{N}$. Thus $\mathsf{succ}\, t = \mathsf{succ}\, n \rightarrow n + 1$, showing that $\mathsf{succ}\, t$ is not stuck.

(T-Pred) Suppose Γ is a context, t_1, t_2 and t_3 are terms, T is a type, $\Gamma \vdash t_1 : \mathsf{Nat}$, $\Gamma \vdash t_2 : \mathsf{Nat} \rightarrow T$ and $\Gamma \vdash t_3 : T$, and assume the inductive hypothesis, $P(\Gamma, t_1, \mathsf{Nat})$, $P(\Gamma, t_2, \mathsf{Nat} \rightarrow T)$ and $P(\Gamma, t_3, T)$. We must show that $P(\Gamma,\ \mathsf{pred}\ t_1\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3,\ T)$. Suppose $\Gamma = \emptyset$. We must show that $\mathsf{pred}\ t_1\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3$ is a closed term that is not stuck. By the inductive hypothesis, we have that t_1, t_2 and t_3 are closed terms that are not stuck. Thus $\mathsf{pred}\ t_1\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3$ is closed. Because t_1 is not stuck, either t_1 is not a normal form, or t_1 is a value, and so there are two cases to consider.

- Suppose t_1 is not a normal form. Then $t_1 \rightarrow t_1'$ for a closed term t_1'. Hence $\mathsf{pred}\ t_1\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3 \rightarrow \mathsf{pred}\ t_1'\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3$, showing that $\mathsf{pred}\ t_1\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3$ is not stuck.
- Suppose t_1 is a value. Since $\emptyset \vdash t_1 : \mathsf{Nat}$, the Canonical Forms Lemma tells us that $t_1 = n$, for some $n \in \mathbb{N}$. There are two subcases to consider.
  - Suppose $n = 0$. Then $\mathsf{pred}\ t_1\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3 = \mathsf{pred}\ 0\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3 \rightarrow t_3$, showing that $\mathsf{pred}\ t_1\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3$ is not stuck.
  - Suppose $n = m + 1$, for some $m \in \mathbb{N}$. Then $\mathsf{pred}\ t_1\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3 = \mathsf{pred}\ m + 1\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3 \rightarrow t_2\, m$, showing that $\mathsf{pred}\ t_1\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3$ is not stuck.

(T-Var) Standard.

(T-Abs) Standard.

(T-App) Standard.

Question C

Let $P(t, t')$ be the predicate between closed terms defined by: $P(t, t')$ iff, for all types T, if $\emptyset \vdash t : T$, then $\emptyset \vdash t' : T$. We use induction on the evaluation relation to show that, for all closed terms t and t', if $t \rightarrow t'$, then $P(t, t')$. (Then, suppose t and t' are closed terms, T is a type, $\emptyset \vdash t : T$ and $t \rightarrow t'$. Thus $P(t, t')$. But $\emptyset \vdash t : T$, and thus $\emptyset \vdash t' : T$.)

(E-SuccNat) Suppose $n \in \mathbb{N}$. We must show that $P(\mathsf{succ}\, n, n+1)$. Suppose T is a type and $\emptyset \vdash \mathsf{succ}\, n : T$. We must show that $\emptyset \vdash n+1 : T$. By the inversion lemma for the typing relation, we have that $T = \mathsf{Nat}$. Thus $\emptyset \vdash n+1 : \mathsf{Nat} = T$.

(E-Succ) Suppose t and t' are closed terms, and $t \rightarrow t'$, and assume the inductive hypothesis, $P(t, t')$. We must show that $P(\mathsf{succ}\, t, \mathsf{succ}\, t')$. Suppose T is a type and $\emptyset \vdash \mathsf{succ}\, t : T$. We must show that $\emptyset \vdash \mathsf{succ}\, t' : T$. By the inversion lemma for the typing relation, we have that $T = \mathsf{Nat}$ and $\emptyset \vdash t : \mathsf{Nat}$. Thus, by the inductive hypothesis, it follows that $\emptyset \vdash t' : \mathsf{Nat}$. Hence $\emptyset \vdash \mathsf{succ}\, t' : \mathsf{Nat} = T$.

(E-PredZero) Suppose t_2 and t_3 are closed terms. We must show that $P(\mathsf{pred}\ 0\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3, t_3)$. Suppose T is a type and $\emptyset \vdash \mathsf{pred}\ 0\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3 : T$. We must show that $\emptyset \vdash t_3 : T$, and this follows by the inversion lemma for the typing relation.

(E-PredNonZero) Suppose $n \in \mathbb{N}$, and t_2 and t_3 are closed terms. We must show that $P(\mathsf{pred}\ n+1\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3, t_2\, n)$. Suppose T is a type and $\emptyset \vdash \mathsf{pred}\ n+1\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3 : T$. We must show that $\emptyset \vdash t_2\, n : T$. By the inversion lemma for the typing relation, we have that $\emptyset \vdash t_2 : \mathsf{Nat} \rightarrow T$. But $\emptyset \vdash n : \mathsf{Nat}$, and thus $\emptyset \vdash t_2\, n : T$.

(E-Pred) Suppose t_1, t_1', t_2 and t_3 are closed terms, and $t_1 \rightarrow t_1'$, and assume the inductive hypothesis, $P(t_1, t_1')$. We must show that $P(\mathsf{pred}\ t_1\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3, \mathsf{pred}\ t_1'\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3)$. Suppose T is a type and $\emptyset \vdash \mathsf{pred}\ t_1\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3 : T$. We must show that $\emptyset \vdash \mathsf{pred}\ t_1'\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3 : T$. By the inversion lemma for the typing relation, we have that $\emptyset \vdash t_1 : \mathsf{Nat}$, $\emptyset \vdash t_2 : \mathsf{Nat} \rightarrow T$ and $\emptyset \vdash t_3 : T$. Since $\emptyset \vdash t_1 : \mathsf{Nat}$, the inductive hypothesis tells us that $\emptyset \vdash t_1' : \mathsf{Nat}$. Since $\emptyset \vdash t_1' : \mathsf{Nat}$, $\emptyset \vdash t_2 : \mathsf{Nat} \rightarrow T$ and $\emptyset \vdash t_3 : T$, it follows that $\emptyset \vdash \mathsf{pred}\ t_1'\ \mathsf{normal}\ t_2\ \mathsf{error}\ t_3 : T$.

(E-App1) Standard.

(E-App2) Standard.

(E-AppAbs) Standard.