CIS 705 — Programming Languages — Spring 2009

Assignment 6

Model Answers

Exercise 1

Let $P = \{\, t \mid t$ is a term and, for all substitutions σ, if $\mathsf{FV}(t) \cap \mathsf{dom}(\sigma) = \emptyset$, then $t\,\sigma = t \,\}$. We use structural induction to show that, for all terms t, $P(t)$.

(Unit Constant) We must show $P(\mathsf{unit})$. Suppose σ is a substitution and $\mathsf{FV}(\mathsf{unit}) \cap \mathsf{dom}(\sigma) = \emptyset$. We have that $\mathsf{unit}\,\sigma = \mathsf{unit}$.

(Pairing) Suppose t_1 and t_2 are terms, and assume the inductive hypothesis: $P(t_1)$ and $P(t_2)$. We must show that $P((t_1, t_2))$. Suppose σ is a substitution and $\mathsf{FV}((t_1, t_2)) \cap \mathsf{dom}(\sigma) = \emptyset$. Because $\mathsf{FV}((t_1, t_2)) = \mathsf{FV}(t_1) \cup \mathsf{FV}(t_2)$, it follows that $\mathsf{FV}(t_1) \cap \mathsf{dom}(\sigma) = \emptyset$ and $\mathsf{FV}(t_2) \cap \mathsf{dom}(\sigma) = \emptyset$. Thus, by the inductive hypothesis, we have that $t_1\,\sigma = t_1$ and $t_2\,\sigma = t_2$, so that $(t_1, t_2)\,\sigma = (t_1\,\sigma, t_2\,\sigma) = (t_1, t_2)$.

(First Projection) Similar to pairing, but slightly simpler.

(Second Projection) Similar to pairing, but slightly simpler.

(Variable) Suppose x is a variable. We must show $P(x)$. Suppose σ is a substitution and $\mathsf{FV}(x) \cap \mathsf{dom}(\sigma) = \emptyset$. Since $\mathsf{FV}(x) = \{x\}$, this means that $x \notin \mathsf{dom}(\sigma)$. Thus $x\,\sigma = x$.

(Abstraction) Suppose x is a variable, T is a type, and t is a term, and assume the inductive hypothesis: $P(t)$. We must show $P(\lambda x : T.\, t)$. Suppose σ is a substitution and $\mathsf{FV}(\lambda x : T.\, t) \cap \mathsf{dom}(\sigma) = \emptyset$. We must show that $(\lambda x : T.\, t)\,\sigma = \lambda x : T.\, t$. Because $(\lambda x : T.\, t)\,\sigma = \lambda x : T.\, t\,(\sigma/\{x\})$, it will suffice to show that $t\,(\sigma/\{x\}) = t$. Thus, by the inductive hypothesis, it will suffice to show that $\mathsf{FV}(t) \cap \mathsf{dom}(\sigma/\{x\}) = \emptyset$. Since $\mathsf{dom}(\sigma/\{x\}) = \mathsf{dom}(\sigma) \setminus \{x\}$, we must show that $\mathsf{FV}(t) \cap (\mathsf{dom}(\sigma) \setminus \{x\}) = \emptyset$. Suppose, toward a contradiction, that $z \in \mathsf{FV}(t)$ and $z \in \mathsf{dom}(\sigma) \setminus \{x\}$. Thus $z \in \mathsf{dom}(\sigma)$ and $z \neq x$. Since $z \in \mathsf{FV}(t)$ and $z \neq x$, we have that $z \in \mathsf{FV}(t) \setminus \{x\} = \mathsf{FV}(\lambda x : T.\, t)$. Hence $z \in \mathsf{FV}(\lambda x : T.\, t)$ and $z \in \mathsf{dom}(\sigma)$, contradicting our assumption that $\mathsf{FV}(\lambda x : T.\, t) \cap \mathsf{dom}(\sigma) = \emptyset$. Thus $\mathsf{FV}(t) \cap (\mathsf{dom}(\sigma) \setminus \{x\}) = \emptyset$, as required.

(Application) Similar to pairing.

Exercise 2

Lemma 2.1
For all sets A, B and C, $(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C$.

Proof. Suppose A, B and C are sets. We must show that $(A \setminus C) \cup (B \setminus C) \subseteq (A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C)$.

- $((A \setminus C) \cup (B \setminus C) \subseteq (A \cup B) \setminus C)$ Suppose $x \in (A \setminus C) \cup (B \setminus C)$. There are two cases to consider.

- Suppose $x \in A \setminus C$, so that $x \in A$ and $x \notin C$. Thus $x \in A \cup B$ and $x \notin C$, so that $x \in (A \cup B) \setminus C$.
- Suppose $x \in B \setminus C$, so that $x \in B$ and $x \notin C$. Thus $x \in A \cup B$ and $x \notin C$, so that $x \in (A \cup B) \setminus C$.

- $((A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C))$ Suppose $x \in (A \cup B) \setminus C$. Thus $x \in A \cup B$ and $x \notin C$. Thus there are two cases to consider.
  - Suppose $x \in A$. Thus $x \in A$ and $x \notin C$, so that $x \in (A \setminus C) \subseteq (A \setminus C) \cup (B \setminus C)$.
  - Suppose $x \in B$. Thus $x \in B$ and $x \notin C$, so that $x \in (B \setminus C) \subseteq (A \setminus C) \cup (B \setminus C)$.

□

Let $P = \{ t \mid t \text{ is a term and, for all substitutions } \sigma, \mathsf{FV}(t\,\sigma) = FV(t) \setminus \mathsf{dom}(\sigma) \}$. We use structural induction to show that, for all terms t, $P(t)$.

(Unit Constant) We must show $P(\mathsf{unit})$. Suppose σ is a substitution. We have that $\mathsf{FV}(\mathsf{unit}\,\sigma) = \mathsf{FV}(\mathsf{unit}) = \emptyset = \emptyset \setminus \mathsf{dom}(\sigma) = \mathsf{FV}(\mathsf{unit}) \setminus \mathsf{dom}(\sigma)$.

(Pairing) Suppose t_1 and t_2 are terms, and assume the inductive hypothesis: $P(t_1)$ and $P(t_2)$. We must show that $P((t_1, t_2))$. Suppose σ is a substitution. We must show that $\mathsf{FV}((t_1, t_2)\,\sigma) = \mathsf{FV}((t_1, t_2)) \setminus \mathsf{dom}(\sigma)$. By the inductive hypothesis and Lemma 2.1, we have that

$$\begin{aligned}
\mathsf{FV}((t_1, t_2)\,\sigma) &= \mathsf{FV}((t_1\,\sigma, t_2\,\sigma)) = \mathsf{FV}(t_1\,\sigma) \cup \mathsf{FV}(t_2\,\sigma) \\
&= (\mathsf{FV}(t_1) \setminus \mathsf{dom}(\sigma)) \cup (\mathsf{FV}(t_2) \setminus \mathsf{dom}(\sigma)) \\
&= (\mathsf{FV}(t_1) \cup \mathsf{FV}(t_2)) \setminus \mathsf{dom}(\sigma) \\
&= \mathsf{FV}((t_1, t_2)) \setminus \mathsf{dom}(\sigma).
\end{aligned}$$

(First Projection) Similar to pairing, but simpler.

(Second Projection) Similar to pairing, but simpler.

(Variable) Suppose x is a variable. We must show that $P(x)$. Suppose σ is a substitution. We must show that $\mathsf{FV}(x\,\sigma) = \mathsf{FV}(x) \setminus \mathsf{dom}(\sigma)$. There are two cases to consider.

- Suppose $x \in \mathsf{dom}(\sigma)$. Then $\mathsf{FV}(x\,\sigma) = \mathsf{FV}(\sigma(x)) = \emptyset = \{x\} \setminus \mathsf{dom}(\sigma) = \mathsf{FV}(x) \setminus \mathsf{dom}(\sigma)$.
- Suppose $x \notin \mathsf{dom}(\sigma)$. Then $\mathsf{FV}(x\,\sigma) = \mathsf{FV}(x) = \{x\} = \{x\} \setminus \mathsf{dom}(\sigma) = \mathsf{FV}(x) \setminus \mathsf{dom}(\sigma)$.

(Abstraction) Suppose x is a variable, T is a type, and t is a term, and assume the inductive hypothesis: $P(t)$. We must show that $P(\lambda x : T.\, t)$. Suppose σ is a substitution. We must show that $\mathsf{FV}((\lambda x : T.\, t)\,\sigma) = \mathsf{FV}(\lambda x : T.\, t) \setminus \mathsf{dom}(\sigma)$. By the inductive hypothesis, we have that

$$\begin{aligned}
\mathsf{FV}((\lambda x : T.\, t)\,\sigma) &= \mathsf{FV}(\lambda x : T.\, t\,(\sigma/\{x\})) \\
&= \mathsf{FV}(t\,(\sigma/\{x\})) \setminus \{x\} \\
&= (\mathsf{FV}(t) \setminus \mathsf{dom}(\sigma/\{x\})) \setminus \{x\}
\end{aligned}$$

$$= (\mathsf{FV}(t) \setminus (\mathsf{dom}(\sigma) \setminus \{x\})) \setminus \{x\}.$$

And $\mathsf{FV}(\lambda x : T.\, t) \setminus \mathsf{dom}(\sigma) = (\mathsf{FV}(t) \setminus \{x\}) \setminus \mathsf{dom}(\sigma)$, so it will suffice to show that

$$(\mathsf{FV}(t) \setminus (\mathsf{dom}(\sigma) \setminus \{x\})) \setminus \{x\} = (\mathsf{FV}(t) \setminus \{x\}) \setminus \mathsf{dom}(\sigma).$$

There are two inclusions to show.

- Suppose $w \in (\mathsf{FV}(t) \setminus (\mathsf{dom}(\sigma) \setminus \{x\})) \setminus \{x\}$. Thus $w \in \mathsf{FV}(t) \setminus (\mathsf{dom}(\sigma) \setminus \{x\})$ and $w \neq x$, so that $w \in \mathsf{FV}(t)$ and $w \notin \mathsf{dom}(\sigma) \setminus \{x\}$. Hence $w \notin \mathsf{dom}(\sigma)$. Because $w \in \mathsf{FV}(t)$ and $w \neq x$, we have that $w \in \mathsf{FV}(t) \setminus \{x\}$. Thus $w \in (\mathsf{FV}(t) \setminus \{x\}) \setminus \mathsf{dom}(\sigma)$.
- Suppose $w \in (\mathsf{FV}(t) \setminus \{x\}) \setminus \mathsf{dom}(\sigma)$. Thus $w \in \mathsf{FV}(t) \setminus \{x\}$ and $w \notin \mathsf{dom}(\sigma)$, so that $w \in \mathsf{FV}(t)$ and $w \neq x$. Because $w \notin \mathsf{dom}(\sigma)$, we have that $w \notin \mathsf{dom}(\sigma) \setminus \{x\}$. Since $w \in \mathsf{FV}(t)$ and $w \notin \mathsf{dom}(\sigma) \setminus \{x\}$, it follows that $w \in \mathsf{FV}(t) \setminus (\mathsf{dom}(\sigma) \setminus \{x\})$. And $w \neq x$, and thus $w \in (\mathsf{FV}(t) \setminus (\mathsf{dom}(\sigma) \setminus \{x\})) \setminus \{x\}$.

(Application) Similar to pairing.

Exercise 3

Lemma 3.1
For all sets A, B and C, $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$.

Proof. Suppose A, B and C are sets. It will suffice to show that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap C) \subseteq A \setminus (B \setminus C)$.

- $(A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap C))$ Suppose $x \in A \setminus (B \setminus C)$, so that $x \in A$ and $x \notin B \setminus C$. There are two subcases to consider.
  - Suppose $x \in C$. Thus $x \in A$ and $x \in C$, so that $x \in A \cap C \subseteq (A \setminus B) \cup (A \cap C)$.
  - Suppose $x \notin C$. Because $x \notin B \setminus C$, it follows that $x \notin B$. Thus $x \in A$ and $x \notin B$, so that $x \in A \setminus B \subseteq (A \setminus B) \cup (A \cap C)$.
- $((A \setminus B) \cup (A \cap C) \subseteq A \setminus (B \setminus C))$ Suppose $x \in (A \setminus B) \cup (A \cap C)$. There are two cases to consider.
  - Suppose $x \in A \setminus B$. Thus $x \in A$ and $x \notin B$, so that $x \notin B \setminus C$. Hence $x \in A \setminus (B \setminus C)$.
  - Suppose $x \in A \cap C$. Thus $x \in A$ and $x \in C$, so that $x \notin B \setminus C$. Hence $x \in A \setminus (B \setminus C)$.

□

Let $P = \{\, (\Gamma, t, T) \mid \Gamma \text{ is a context}, t \text{ is a term}, T \text{ is a type and, for all substitutions } \sigma,\ \text{if, for all } x \in \mathsf{dom}(\Gamma) \cap \mathsf{dom}(\sigma),\ \emptyset \vdash \sigma(x) : \Gamma(x),\ \text{then } \Gamma/\mathsf{dom}(\sigma) \vdash t\,\sigma : T \,\}$. We use induction on the typing relation to prove that, for all contexts Γ, terms t and types T, if $\Gamma \vdash t : T$, then $P(\Gamma, t, T)$.

(T-Unit) Suppose Γ is a context. We must show that $P(\Gamma, \mathsf{unit}, \mathsf{Unit})$. Suppose σ is a substitution, and for all $x \in \mathsf{dom}(\Gamma) \cap \mathsf{dom}(\sigma)$, $\emptyset \vdash \sigma(x) : \Gamma(x)$. We must show that $\Gamma/\mathsf{dom}(\sigma) \vdash \mathsf{unit}\,\sigma : \mathsf{Unit}$. And this follows by (T-Unit), since $\mathsf{unit}\,\sigma = \mathsf{unit}$.

(T-Pair) Suppose Γ is a context, t_1 and t_2 are terms, T_1 and T_2 are types, $\Gamma \vdash t_1 : T_1$ and $\Gamma \vdash t_2 : T_2$, and assume the inductive hypothesis, $P(\Gamma, t_1, T_1)$ and $P(\Gamma, t_2, T_2)$. We must show that $P(\Gamma, (t_1, t_2), T_1 \times T_2)$. Suppose σ is a substitution, and for all $x \in \mathsf{dom}(\Gamma) \cap \mathsf{dom}(\sigma)$, $\emptyset \vdash \sigma(x) : \Gamma(x)$. We must show that $\Gamma/\mathsf{dom}(\sigma) \vdash (t_1, t_2)\,\sigma : T_1 \times T_2$. Because of $P(\Gamma, t_1, T_1)$, we have that $\Gamma/\mathsf{dom}(\sigma) \vdash t_1\,\sigma : T_1$, and because of $P(\Gamma, t_2, T_2)$, we have that $\Gamma/\mathsf{dom}(\sigma) \vdash t_2\,\sigma : T_2$. Thus $\Gamma/\mathsf{dom}(\sigma) \vdash (t_1\,\sigma, t_2\,\sigma) : T_1 \times T_2$, by (T-Pair). But $(t_1, t_2)\,\sigma = (t_1\,\sigma, t_2\,\sigma)$, and so $\Gamma/\mathsf{dom}(\sigma) \vdash (t_1, t_2)\,\sigma : T_1 \times T_2$.

(T-Fst) Similar to (T-Pair), but slightly simpler.

(T-Snd) Similar to (T-Pair), but slightly simpler.

(T-Var) Suppose Γ is a context, x is a variable, T is a type and $(x, T) \in \Gamma$. We must show that $P(\Gamma, x, T)$. Suppose σ is a substitution, and for all $x \in \mathsf{dom}(\Gamma) \cap \mathsf{dom}(\sigma)$, $\emptyset \vdash \sigma(x) : \Gamma(x)$. We must show that $\Gamma/\mathsf{dom}(\sigma) \vdash x\,\sigma : T$. There are two cases to consider.

- Suppose $x \in \mathsf{dom}(\sigma)$. Since $(x, T) \in \Gamma$, we have that $x \in \mathsf{dom}(\Gamma) \cap \mathsf{dom}(\sigma)$. Thus by our assumption we have that $\emptyset \vdash \sigma(x) : \Gamma(x)$. But $\sigma(x) = x\,\sigma$ and $\Gamma(x) = T$, so that $\emptyset \vdash x\,\sigma : T$. Thus the weakening lemma tells us that $\Gamma/\mathsf{dom}(\sigma) \vdash x\,\sigma : T$.
- Suppose $x \notin \mathsf{dom}(\sigma)$. Then $x\,\sigma = x$, so we must show that $\Gamma/\mathsf{dom}(\sigma) \vdash x : T$. By (T-Var), it will suffice to show that $(x, T) \in \Gamma/\mathsf{dom}(\sigma)$. Since $\mathsf{dom}(\Gamma/\mathsf{dom}(\sigma)) = \mathsf{dom}(\Gamma) \setminus \mathsf{dom}(\sigma)$, $x \in \mathsf{dom}(\Gamma)$ and $x \notin \mathsf{dom}(\sigma)$, we have that $x \in \mathsf{dom}(\Gamma/\mathsf{dom}(\sigma))$. Thus $(\Gamma/\mathsf{dom}(\sigma))(x) = \Gamma(x) = T$, since $(x, T) \in \Gamma$, showing that $(x, T) \in \Gamma/\mathsf{dom}(\sigma)$.

(T-Abs) Suppose Γ is a context, x is a variable, t is a term, T_1 and T_2 are types, and $\Gamma[x \mapsto T_1] \vdash t : T_2$, and assume the inductive hypothesis, $P(\Gamma[x \mapsto T_1], t, T_2)$. We must show that $P(\Gamma, \lambda x : T_1 .\, t, T_1 \to T_2)$. Suppose σ is a substitution, and, (†) for all $x \in \mathsf{dom}(\Gamma) \cap \mathsf{dom}(\sigma)$, $\emptyset \vdash \sigma(x) : \Gamma(x)$. We must show that $\Gamma/\mathsf{dom}(\sigma) \vdash (\lambda x : T_1 .\, t)\,\sigma : T_1 \to T_2$. Since $(\lambda x : T_1 .\, t)\,\sigma = \lambda x : T_1 .\, t\,(\sigma/\{x\})$, we must show that $\Gamma/\mathsf{dom}(\sigma) \vdash \lambda x : T_1 .\, t\,(\sigma/\{x\}) : T_1 \to T_2$. And, by T-Abs, it will suffice to show that $(\Gamma/\mathsf{dom}(\sigma))[x \mapsto T_1] \vdash t\,(\sigma/\{x\}) : T_2$.

Next, we show that, (‡) for all $y \in \mathsf{dom}(\Gamma[x \mapsto T_1]) \cap \mathsf{dom}(\sigma/\{x\})$, $\emptyset \vdash (\sigma/\{x\})(y) : \Gamma[x \mapsto T_1](y)$. Suppose $y \in \mathsf{dom}(\Gamma[x \mapsto T_1]) \cap \mathsf{dom}(\sigma/\{x\})$. Thus $y \in \mathsf{dom}(\Gamma[x \mapsto T_1]) = \mathsf{dom}(\Gamma) \cup \{x\}$ and $y \in \mathsf{dom}(\sigma/\{x\}) = \mathsf{dom}(\sigma) \setminus \{x\}$, so that $y \in \mathsf{dom}(\sigma)$ and $y \neq x$. Thus $y \in \mathsf{dom}(\Gamma) \cap \mathsf{dom}(\sigma)$, so that (†) tells us that $\emptyset \vdash \sigma(y) : \Gamma(y)$. We have that $\sigma(y) = (\sigma/\{x\})(y)$ and (because $(y \neq x)$ $\Gamma(y) = \Gamma[x \mapsto T_1](y)$. Thus $\emptyset \vdash (\sigma/\{x\})(y) : \Gamma[x \mapsto T_1](y)$. This completes the proof of (‡).

Because of (‡) and the inductive hypothesis, $P(\Gamma[x \mapsto T_1], t, T_2)$, we have that

$$\Gamma[x \mapsto T_1]/\mathsf{dom}(\sigma/\{x\}) \vdash t\,(\sigma/\{x\}) : T_2.$$

Thus it will suffice to show that $(\Gamma/\mathsf{dom}(\sigma))[x \mapsto T_1] = \Gamma[x \mapsto T_1]/\mathsf{dom}(\sigma/\{x\})$. By Lemma 3.1, we have that

$$\begin{aligned}
\mathsf{dom}((\Gamma/\mathsf{dom}(\sigma))[x \mapsto T_1]) &= \mathsf{dom}(\Gamma/\mathsf{dom}(\sigma)) \cup \{x\} \\
&= (\mathsf{dom}(\Gamma) \setminus \mathsf{dom}(\sigma)) \cup \{x\} \\
&= ((\mathsf{dom}(\Gamma) \cup \{x\}) \setminus \mathsf{dom}(\sigma)) \cup \{x\} \\
&= ((\mathsf{dom}(\Gamma) \cup \{x\}) \setminus \mathsf{dom}(\sigma)) \cup ((\mathsf{dom}(\Gamma) \cup \{x\}) \cap \{x\})
\end{aligned}$$

$$= (\mathsf{dom}(\Gamma) \cup \{x\}) \setminus (\mathsf{dom}(\sigma) \setminus \{x\})$$
$$= \mathsf{dom}(\Gamma[x \mapsto T_1]) \setminus \mathsf{dom}(\sigma/\{x\})$$
$$= \mathsf{dom}(\Gamma[x \mapsto T_1]/(\sigma/\{x\})).$$

Finally, suppose $y \in \mathsf{dom}((\Gamma/\mathsf{dom}(\sigma))[x \mapsto T_1]) = \mathsf{dom}(\Gamma[x \mapsto T_1]/(\sigma/\{x\}))$. We must show that $(\Gamma/\mathsf{dom}(\sigma))[x \mapsto T_1](y) = (\Gamma[x \mapsto T_1]/(\sigma/\{x\}))(y)$. There are two cases to consider.

- Suppose $y = x$. Then $(\Gamma/\mathsf{dom}(\sigma))[x \mapsto T_1](y) = T_1 = \Gamma[x \mapsto T_1](y) = (\Gamma[x \mapsto T_1]/(\sigma/\{x\}))(y)$.
- Suppose $y \neq x$. Then $(\Gamma/\mathsf{dom}(\sigma))[x \mapsto T_1](y) = (\Gamma/\mathsf{dom}(\sigma))(y) = \Gamma(y) = \Gamma[x \mapsto T_1](y) = (\Gamma[x \mapsto T_1]/(\sigma/\{x\}))(y)$.

(T-App) Similar to (T-Pair).

Exercise 4

Let $P = \{\, T \mid T$ is a type and, for all closed terms t and t', if $\emptyset \vdash t : T$ and $t \rightarrow t'$, then $R_T(t)$ iff $R_T(t')\,\}$. We use structural induction to show that, for all types T, $P(T)$.

(Unit Type) We must show $P(\mathsf{Unit})$. Suppose t and t' are closed terms, $\emptyset \vdash t : \mathsf{Unit}$ and $t \rightarrow t'$. We must show $R_{\mathsf{Unit}}(t)$ iff $R_{\mathsf{Unit}}(t')$. By preservation, we know that $\emptyset \vdash t' : \mathsf{Unit}$. There are two directions to show.

- Suppose $R_{\mathsf{Unit}}(t)$. Thus t converges, so that $t \rightarrow^* v$ for some closed value v. By the determinacy of the evaluation relation, it follows that $t' \rightarrow^* v$, showing the t' converges. And $\emptyset \vdash t' : \mathsf{Unit}$, so that $R_{\mathsf{Unit}}(t')$.
- Suppose $R_{\mathsf{Unit}}(t')$. Thus t' converges, so that $t' \rightarrow^* v$ for some closed value v. Thus $t \rightarrow^* v$, showing the t converges. And $\emptyset \vdash t : \mathsf{Unit}$, so that $R_{\mathsf{Unit}}(t)$.

(Product Type) Suppose T_1 and T_2 are types, and assume the inductive hypothesis, $P(T_1)$ and $P(T_2)$. We must show that $P(T_1 \times T_2)$. Suppose t and t' are closed terms, $\emptyset \vdash t : T_1 \times T_2$ and $t \rightarrow t'$. We must show $R_{T_1 \times T_2}(t)$ iff $R_{T_1 \times T_2}(t')$. By preservation, we know that $\emptyset \vdash t' : T_1 \times T_2$. There are two directions to show.

- Suppose $R_{T_1 \times T_2}(t)$. Thus t converges, $R_{T_1}(\mathsf{fst}\, t)$ and $R_{T_2}(\mathsf{snd}\, t)$, so that $t \rightarrow^* v$ for some closed value v, $\emptyset \vdash \mathsf{fst}\, t : T_1$ and $\emptyset \vdash \mathsf{snd}\, t : T_2$. By the determinacy of the evaluation relation, it follows that $t' \rightarrow^* v$, showing the t' converges. Because $t \rightarrow t'$, we have that $\mathsf{fst}\, t \rightarrow \mathsf{fst}\, t'$. And $R_{T_1}(\mathsf{fst}\, t)$, so that $P(T_1)$ tells us that $R_{T_1}(\mathsf{fst}\, t')$. Because $t \rightarrow t'$, we have that $\mathsf{snd}\, t \rightarrow \mathsf{snd}\, t'$. And $R_{T_2}(\mathsf{snd}\, t)$, so that $P(T_2)$ tells us that $R_{T_2}(\mathsf{snd}\, t')$. Since $\emptyset \vdash t' : T_1 \times T_2$, t' converges, $R_{T_1}(\mathsf{fst}\, t')$ and $R_{T_2}(\mathsf{snd}\, t')$, we can conclude that $R_{T_1 \times T_2}(t')$.
- Suppose $R_{T_1 \times T_2}(t')$. Thus t' converges, $R_{T_1}(\mathsf{fst}\, t')$ and $R_{T_2}(\mathsf{snd}\, t')$, so that $t' \rightarrow^* v$ for some closed value v. Thus $t \rightarrow^* v$, showing the t converges. Since $\emptyset \vdash t : T_1 \times T_2$, we have that $\emptyset \vdash \mathsf{fst}\, t : T_1$ and $\emptyset \vdash \mathsf{snd}\, t : T_2$. Because $t \rightarrow t'$, we have that $\mathsf{fst}\, t \rightarrow \mathsf{fst}\, t'$. And $R_{T_1}(\mathsf{fst}\, t')$, so that $P(T_1)$ tells us that $R_{T_1}(\mathsf{fst}\, t)$. Because $t \rightarrow t'$, we have that $\mathsf{snd}\, t \rightarrow \mathsf{snd}\, t'$. And $R_{T_2}(\mathsf{snd}\, t')$, so that $P(T_2)$ tells us that $R_{T_2}(\mathsf{snd}\, t)$. Since $\emptyset \vdash t : T_1 \times T_2$, t converges, $R_{T_1}(\mathsf{fst}\, t)$ and $R_{T_2}(\mathsf{snd}\, t)$, we can conclude that $R_{T_1 \times T_2}(t)$.

(Function Type) Suppose T_1 and T_2 are types, and assume the inductive hypothesis, $P(T_1)$ and $P(T_2)$. We must show that $P(T_1 \to T_2)$. Suppose t and t' are closed terms, $\emptyset \vdash t : T_1 \to T_2$ and $t \to t'$. We must show $R_{T_1 \to T_2}(t)$ iff $R_{T_1 \to T_2}(t')$. By preservation, we know that $\emptyset \vdash t' : T_1 \to T_2$. There are two directions to show.

- Suppose $R_{T_1 \to T_2}(t)$. Thus t converges, and, (†) for all terms s, if $R_{T_1}(s)$, then $R_{T_2}(t\,s)$. Because t converges, we have that $t \to^* v$ for some closed value v. By the determinacy of the evaluation relation, it follows that $t' \to^* v$, showing the t' converges. Because $\emptyset \vdash t' : T_1 \to T_2$ and t' converges, to show that $R_{T_1 \to T_2}(t')$, it remains to show that, for all terms s, if $R_{T_1}(s)$, then $R_{T_2}(t'\,s)$. Suppose s is a term and $R_{T_1}(s)$. We must show that $R_{T_2}(t'\,s)$. Since $R_{T_1}(s)$, we know that $\emptyset \vdash s : T_1$, so that $\emptyset \vdash t\,s : T_2$ and s is closed. And, from (†) and $R_{T_1}(s)$, we can conclude $R_{T_2}(t\,s)$. Since $t \to t'$, it follows that $t\,s \to t'\,s$. Thus $P(T_2)$ tells us that $R_{T_2}(t'\,s)$, as required.
- Suppose $R_{T_1 \to T_2}(t')$. Thus t' converges, and, (†) for all terms s, if $R_{T_1}(s)$, then $R_{T_2}(t'\,s)$. Because t' converges, we have that $t' \to^* v$ for some closed value v. Thus $t \to^* v$, showing the t converges. Because $\emptyset \vdash t : T_1 \to T_2$ and t converges, to show that $R_{T_1 \to T_2}(t)$, it remains to show that, for all terms s, if $R_{T_1}(s)$, then $R_{T_2}(t\,s)$. Suppose s is a term and $R_{T_1}(s)$. We must show that $R_{T_2}(t\,s)$. Since $R_{T_1}(s)$, we know that $\emptyset \vdash s : T_1$, so that $\emptyset \vdash t\,s : T_2$ and s is closed. And, from (†) and $R_{T_1}(s)$, we can conclude $R_{T_2}(t'\,s)$. Since $t \to t'$, it follows that $t\,s \to t'\,s$. Thus $P(T_2)$ tells us that $R_{T_2}(t\,s)$, as required.

Exercise 5

Lemma 5.1
For all functions f and sets A, $(f/A)/A = f/A$.

Proof. Suppose f is a function and A is a set. Since $\mathsf{dom}((f/A)/A) = \mathsf{dom}(f/A) \setminus A = (\mathsf{dom}(f) \setminus A) \setminus A = \mathsf{dom}(f) \setminus A = \mathsf{dom}(f/A)$, we have that $(f/A)/A$ and f/A have the same domain, $\mathsf{dom}(f) \setminus A$. And, for all x in that common domain, $((f/A)/A)(x) = (f/A)(x) = f(x) = (f/A)(x)$, completing the proof that $(f/A)/A = f/A$. □

Lemma 5.2
For all functions f and sets A and B, $(f/A)/B = (f/B)/A$.

Proof. Suppose f is a function and A and B are sets. Since $\mathsf{dom}((f/A)/B) = \mathsf{dom}(f/A) \setminus B = (\mathsf{dom}(f) \setminus A) \setminus B = (\mathsf{dom}(f) \setminus B) \setminus A = \mathsf{dom}(f/B) \setminus A = \mathsf{dom}((f/B)/A)$, we have that $(f/A)/B$ and $(f/B)/A$ have the same domain $(\mathsf{dom}(f) \setminus A) \setminus B = (\mathsf{dom}(f) \setminus B) \setminus A$. And, for all x in that common domain, we have that $((f/A)/B)(x) = (f/A)(x) = f(x) = (f/B)(x) = ((f/B)/A)(x)$, completing the proof that $(f/A)/B = (f/B)/A$. □

Lemma 5.3
For all functions f and elements x and y of our universe, $f[x \mapsto y]/\{x\} = f/\{x\}$.

Proof. Suppose f is a function and x are y are elements of our universe. Since $\mathsf{dom}(f[x \mapsto y]/\{x\}) = \mathsf{dom}(f[x \mapsto y]) \setminus \{x\} = (\mathsf{dom}(f) \cup \{x\}) \setminus \{x\} = \mathsf{dom}(f) \setminus \{x\} = \mathsf{dom}(f/\{x\})$, we have that $f[x \mapsto y]/\{x\}$ and $f/\{x\}$ have the same domain, $\mathsf{dom}(f) \setminus \{x\}$. And, for all $z \in \mathsf{dom}(f) \setminus \{x\}$,

$(f[x \mapsto y]/\{x\})(z) = f[x \mapsto y](z) = f(z) = (f/\{x\})(z)$, completing the proof that $f[x \mapsto y]/\{x\} = f/\{x\}$. $\square$

Lemma 5.4
For all functions f and elements x, y and z of our universe, if $x \neq y$, then $(f/\{y\})[x \mapsto z] = f[x \mapsto z]/\{y\}$.

Proof. Suppose f is a function, x, y and z are elements of our universe, and $x \neq y$. We have that $\mathsf{dom}((f/\{y\})[x \mapsto z]) = \mathsf{dom}(f/\{y\}) \cup \{x\} = (\mathsf{dom}(f) \setminus \{y\}) \cup \{x\} = (\mathsf{dom}(f) \cup \{x\}) \setminus \{y\} = \mathsf{dom}(f[x \mapsto z]) \setminus \{y\} = \mathsf{dom}(f[x \mapsto z]/\{y\})$. Suppose w is in the common domain of $(f/\{y\})[x \mapsto z]$ and $f[x \mapsto z]/\{y\}$. We must show that $((f/\{y\})[x \mapsto z])(w) = (f[x \mapsto z]/\{y\})(w)$. There are two cases to consider.

- Suppose $w = x$. Then $((f/\{y\})[x \mapsto z])(w) = z = f[x \mapsto z](w) = (f[x \mapsto z]/\{y\})(w)$.
- Suppose $w \neq x$. Then $((f/\{y\})[x \mapsto z])(w) = (f/\{y\})(w) = f(w) = f[x \mapsto z](w) = (f[x \mapsto z]/\{y\})(w)$.

$\square$

Lemma 5.5
For all terms t, closed values v, variables x and substitutions σ,

$$(t\,(\sigma/\{x\}))\,\{(x, v)\} = t\,\sigma[x \mapsto v].$$

Proof. Suppose v is a closed value and x is a variable. Let $P = \{\, t \mid t$ is a term, and, for all substitutions $\sigma, (t\,(\sigma/\{x\}))\,\{(x, v)\} = t\,\sigma[x \mapsto v]\,\}$. We use structural induction to prove that, for all terms t, $P(t)$.

(Unit Constant) We must show $P(\mathsf{unit})$. Suppose σ is a substitution. We have that $(\mathsf{unit}\,(\sigma/\{x\}))\,\{(x, v)\} = \mathsf{unit}\,\{(x, v)\} = \mathsf{unit} = \mathsf{unit}\,\sigma[x \mapsto v]$.

(Pairing) Suppose t_1 and t_2 are terms, and assume the inductive hypothesis: $P(t_1)$ and $P(t_2)$. We must show that $P((t_1, t_2))$. Suppose σ is a substitution. By the inductive hypothesis, we have that

$$\begin{aligned}
((t_1, t_2)\,(\sigma/\{x\}))\,\{(x, v)\} &= (t_1\,(\sigma/\{x\}), t_2\,(\sigma/\{x\}))\,\{(x, v)\} \\
&= ((t_1\,(\sigma/\{x\}))\,\{(x, v)\}, (t_2\,(\sigma/\{x\}))\,\{(x, v)\}) \\
&= (t_1\,\sigma[x \mapsto v], t_2\,\sigma[x \mapsto v]) \\
&= (t_1, t_2)\,\sigma[x \mapsto v].
\end{aligned}$$

(First Projection) Similar to pairing.

(Second Projection) Similar to pairing.

(Variable) Suppose y is a variable. We must show $P(y)$. Suppose σ is a substitution. There are two cases to consider.

- Suppose $y = x$. Then $y \notin \mathsf{dom}(\sigma) \setminus \{x\} = \mathsf{dom}(\sigma/\{x\})$, $y \in \mathsf{dom}\{(x,v)\}$ and $y \in \mathsf{dom}(\sigma) \cup \{x\} = \mathsf{dom}(\sigma[x \mapsto v])$, so that $(y\,(\sigma/\{x\}))\,\{(x,v)\} = y\,\{(x,v)\} = \{(x,v)\}(y) = v = \sigma[x \mapsto v](y) = y\,\sigma[x \mapsto v]$.
- Suppose $y \neq x$. There are two sub-cases to consider
  - Suppose $y \in \mathsf{dom}(\sigma)$. Since $y \in \mathsf{dom}(\sigma) \setminus \{x\} = \mathsf{dom}(\sigma/\{x\})$, we have that $(y\,(\sigma/\{x\}))\,\{(x,v)\} = ((\sigma/\{x\})(y))\,\{(x,v)\} = (\sigma(y))\,\{(x,v)\}$. But $\sigma(y)$ is closed, so that $\mathsf{FV}(\sigma(y)) \cap \mathsf{dom}(\{(x,v)\}) = \emptyset$, and thus Exercise 1 tells us that $(\sigma(y))\,\{(x,v)\} = \sigma(y)$. Furthermore, because $y \in \mathsf{dom}(\sigma) \cup \{x\} = \mathsf{dom}(\sigma[x \mapsto v])$, we have that $\sigma(y) = \sigma[x \mapsto v](y) = y\,\sigma[x \mapsto v]$. Hence $(y\,(\sigma/\{x\}))\,\{(x,v)\} = (\sigma(y))\,\{(x,v)\} = \sigma(y) = y\,\sigma[x \mapsto v]$.
  - Suppose $y \notin \mathsf{dom}(\sigma)$. Because $y \notin \mathsf{dom}(\sigma) \setminus \{x\} = \mathsf{dom}(\sigma/\{x\})$, $y \notin \mathsf{dom}\{(x,v)\}$ and $y \notin \mathsf{dom}(\sigma) \cup \{x\} = \mathsf{dom}(\sigma[x \mapsto v])$, we have that $(y\,(\sigma/\{x\}))\,\{(x,v)\} = y\,\{(x,v)\} = y = y\,\sigma[x \mapsto v]$.

(Abstraction) Suppose y is a variable, T is a type, and t is a term, and assume the inductive hypothesis: $P(t)$. We must show $P(\lambda y : T.\, t)$. Suppose σ is a substitution. There are two cases to consider.

- Suppose $y = x$. By Lemma 5.1 and since $\{(y,v)\}/\{y\} = \emptyset$ ($\mathsf{dom}(\{(y,v)\}/\{y\}) = \mathsf{dom}(\{(y,v)\} \setminus \{y\} = \{y\} \setminus \{y\} = \emptyset$), $(t\,(\sigma/\{y\}))\,\emptyset = t\,(\sigma/\{y\})$ (by Exercise 1, since $\mathsf{FV}(t\,(\sigma/\{y\})) \cap \mathsf{dom}(\emptyset) = \mathsf{FV}(t\,(\sigma/\{y\})) \cap \emptyset = \emptyset$) and $\sigma/\{y\} = \sigma/\{x\} = \sigma[x \mapsto v]/\{x\} = \sigma[x \mapsto v]/\{y\}$ (by Lemma 5.3), we have that

$$\begin{aligned}
((\lambda y : T.\, t)(\sigma/\{x\}))\,\{(x,v)\} &= (\lambda y : T.\, t\,((\sigma/\{x\})/\{y\}))\,\{(x,v)\}\\
&= (\lambda y : T.\, t\,((\sigma/\{y\})/\{y\}))\,\{(x,v)\}\\
&= (\lambda y : T.\, t\,(\sigma/\{y\}))\,\{(x,v)\}\\
&= \lambda y : T.\, (t\,(\sigma/\{y\}))\,(\{(x,v)\}/\{y\})\\
&= \lambda y : T.\, (t\,(\sigma/\{y\}))\,(\{(y,v)\}/\{y\})\\
&= \lambda y : T.\, (t\,(\sigma/\{y\}))\,\emptyset\\
&= \lambda y : T.\, t\,(\sigma/\{y\})\\
&= \lambda y : T.\, t\,(\sigma[x \mapsto v]/\{y\})\\
&= (\lambda y : T.t)\,\sigma[x \mapsto v].
\end{aligned}$$

- Suppose $y \neq x$. By Lemmas 5.2 and 5.4, the inductive hypothesis, and since $(\{(x,v)\}/y) = \{(x,v)\}$, we have that

$$\begin{aligned}
((\lambda y : T.\, t)(\sigma/\{x\}))\,\{(x,v)\} &= (\lambda y : T.\, t\,((\sigma/\{x\})/\{y\}))\,\{(x,v)\}\\
&= \lambda y : T.\, (t\,((\sigma/\{x\})/\{y\}))\,(\{(x,v)\}/y)\\
&= \lambda y : T.\, (t\,((\sigma/\{x\})/\{y\}))\,\{(x,v)\}\\
&= \lambda y : T.\, (t\,((\sigma/\{y\})/\{x\}))\,\{(x,v)\}\\
&= \lambda y : T.\, t\,((\sigma/\{y\})[x \mapsto v])\\
&= \lambda y : T.\, t\,(\sigma[x \mapsto v]/\{y\})\\
&= (\lambda y : T.t)\,\sigma[x \mapsto v].
\end{aligned}$$

(Application) Similar to pairing.

□

Let $P = \{(\Gamma, t, T) \mid \Gamma$ is a context, t is a term, T is a type and, for all substitutions σ, if $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(\sigma)$ and, for all $x \in \mathsf{dom}(\Gamma), R_{\Gamma(x)}(\sigma(x))$, then $R_T(t\,\sigma)\,\}$. We use induction on the typing relation to prove that, for all contexts Γ, terms t and types T, if $\Gamma \vdash t : T$, then $P(\Gamma, t, T)$.

(T-Unit) Suppose Γ is a context. We must show that $P(\Gamma, \mathsf{unit}, \mathsf{Unit})$. Suppose σ is a substitution, $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(\sigma)$, and for all $x \in \mathsf{dom}(\Gamma)$, $R_{\Gamma(x)}(\sigma(x))$. We must show that $R_{\mathsf{Unit}}(\mathsf{unit}\,\sigma)$. Since $\mathsf{unit}\,\sigma = \mathsf{unit}$, it will suffice to show that $R_{\mathsf{Unit}}(\mathsf{unit})$, and this follows since $\emptyset \vdash \mathsf{unit} : \mathsf{Unit}$ (by (T-Unit)) and unit is a value and thus converges.

(T-Pair) Suppose Γ is a context, t_1 and t_2 are terms, T_1 and T_2 are types, $\Gamma \vdash t_1 : T_1$ and $\Gamma \vdash t_2 : T_2$, and assume the inductive hypothesis, $P(\Gamma, t_1, T_1)$ and $P(\Gamma, t_2, T_2)$. We must show that $P(\Gamma, (t_1, t_2), T_1 \times T_2)$. Suppose σ is a substitution, $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(\sigma)$, and for all $x \in \mathsf{dom}(\Gamma)$, $R_{\Gamma(x)}(\sigma(x))$. We must show that $R_{T_1 \times T_2}((t_1, t_2)\,\sigma)$. Since $(t_1, t_2)\,\sigma = (t_1\,\sigma, t_2\,\sigma)$, it will suffice to show $R_{T_1 \times T_2}((t_1\,\sigma, t_2\,\sigma))$. By the inductive hypothesis, we have that $R_{T_1}(t_1\,\sigma)$ and $R_{T_2}(t_2\,\sigma)$. Since $R_{T_1}(t_1\,\sigma)$, we have that $\emptyset \vdash t_1\,\sigma : T_1$ and $t_1\,\sigma$ converges, so that $t_1\,\sigma \rightarrow^* v_1$, for some closed value v_1. By Exercise 4, it follows that $R_{T_1}(v_1)$. And, since $R_{T_2}(t_2\,\sigma)$, we have that $\emptyset \vdash t_2\,\sigma : T_2$ and $t_2\,\sigma$ converges, so that $t_2\,\sigma \rightarrow^* v_2$, for some closed value v_2. By Exercise 4, it follows that $R_{T_2}(v_2)$. Since $\emptyset \vdash t_1\,\sigma : T_1$ and $\emptyset \vdash t_2\,\sigma : T_2$, (T-Pair) gives us $\emptyset \vdash (t_1\,\sigma, t_2\,\sigma) : T_1 \times T_2$. Since $t_1\,\sigma \rightarrow^* v_1$ and $t_2\,\sigma \rightarrow^* v_2$, we have that $(t_1\,\sigma, t_2\,\sigma) \rightarrow^* (v_1, t_2\,\sigma) \rightarrow^* (v_1, v_2)$. Thus, by Exercise 4, it will suffice to show that $R_{T_1 \times T_2}((v_1, v_2))$. By the preservation theorem, we have that $\emptyset \vdash (v_1, v_2) : T_1 \times T_2$. Since (v_1, v_2) is a value, it converges. It remains to show that $R_{T_1}(\mathsf{fst}\,(v_1, v_2))$ and $R_{T_2}(\mathsf{snd}\,(v_1, v_2))$. Since $\emptyset \vdash (v_1, v_2) : T_1 \times T_2$, we have that $\emptyset \vdash \mathsf{fst}\,(v_1, v_2) : T_1$ and $\emptyset \vdash \mathsf{snd}\,(v_1, v_2) : T_2$. Furthermore, $\mathsf{fst}\,(v_1, v_2) \rightarrow v_1$ and $\mathsf{snd}\,(v_1, v_2) \rightarrow v_2$. Thus, since $R_{T_1}(v_1)$ and $R_{T_2}(v_2)$, Exercise 4 tells us that $R_{T_1}(\mathsf{fst}\,(v_1, v_2))$ and $R_{T_2}(\mathsf{snd}\,(v_1, v_2))$.

(T-Fst) Suppose Γ is a context, t is a term, T_1 and T_2 are types, $\Gamma \vdash t : T_1 \times T_2$, and assume the inductive hypothesis, $P(\Gamma, t, T_1 \times T_2)$. We must show that $P(\Gamma, \mathsf{fst}\,t, T_1)$. Suppose σ is a substitution, $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(\sigma)$, and for all $x \in \mathsf{dom}(\Gamma)$, $R_{\Gamma(x)}(\sigma(x))$. We must show that $R_{T_1}((\mathsf{fst}\,t)\,\sigma)$. Since $(\mathsf{fst}\,t)\,\sigma = \mathsf{fst}(t\,\sigma)$, it will suffice to show that $R_{T_1}(\mathsf{fst}(t\,\sigma))$. By the inductive hypothesis, we have that $R_{T_1 \times T_2}(t\,\sigma)$. Thus $R_{T_1}(\mathsf{fst}(t\,\sigma))$.

(T-Snd) Similar to (T-Fst).

(T-Var) Suppose Γ is a context, x is a variable, T is a type and $(x, T) \in \Gamma$. We must show that $P(\Gamma, x, T)$. Suppose σ is a substitution, $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(\sigma)$, and for all $x \in \mathsf{dom}(\Gamma)$, $R_{\Gamma(x)}(\sigma(x))$. We must show that $R_T(x\,\sigma)$. Since $x \in \mathsf{dom}(\Gamma)$, we have that $R_{\Gamma(x)}(\sigma(x))$. Since $(x, T) \in \Gamma$, we have that $\Gamma(x) = T$. And, because $x \in \mathsf{dom}(\Gamma)$, we have that $x \in \mathsf{dom}(\sigma)$, so that $\sigma(x) = x\,\sigma$. Thus $R_T(x\,\sigma))$.

(T-Abs) Suppose Γ is a context, x is a variable, t is a term, T_1 and T_2 are types, and $\Gamma[x \mapsto T_1] \vdash t : T_2$, and assume the inductive hypothesis, $P(\Gamma[x \mapsto T_1], t, T_2)$. We must show that $P(\Gamma, \lambda x : T_1.\,t, T_1 \rightarrow T_2)$. Suppose σ is a substitution, $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(\sigma)$, and for all $x \in \mathsf{dom}(\Gamma)$, $R_{\Gamma(x)}(\sigma(x))$. We must show that $R_{T_1 \rightarrow T_2}((\lambda x : T_1.\,t)\,\sigma)$. Since $(\lambda x : T_1.\,t)\,\sigma = \lambda x : T_1.\,t\,(\sigma/\{x\})$, it will suffice to show $R_{T_1 \rightarrow T_2}(\lambda x : T_1.\,t\,(\sigma/\{x\}))$.

Because $\Gamma[x \mapsto T_1] \vdash t : T_2$, (T-Abs) gives us $\Gamma \vdash \lambda x : T_1.\, t : T_1 \to T_2$. Since $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(\sigma)$, and for all $x \in \mathsf{dom}(\Gamma)$, $R_{\Gamma(x)}(\sigma(x))$, we have that, for all $x \in \mathsf{dom}(\Gamma) \cap \mathsf{dom}(\sigma)$, $\emptyset \vdash \sigma(x) : \Gamma(x)$. Thus Exercise 3 tells us that $\Gamma/\mathsf{dom}(\sigma) \vdash (\lambda x : T_1.\, t)\, \sigma : T_1 \to T_2$. But $\Gamma/\mathsf{dom}(\sigma) = \emptyset$ and $(\lambda x : T_1.\, t)\, \sigma = \lambda x : T_1.\, t\, (\sigma/\{x\})$, and thus $\emptyset \vdash \lambda x : T_1.\, t\, (\sigma/\{x\}) : T_1 \to T_2$. Furthermore, $\lambda x : T_1.\, t\, (\sigma/\{x\})$ is a value, and thus converges. So to show $R_{T_1 \to T_2}(\lambda x : T_1.\, t\, (\sigma/\{x\}))$, it remains to show that, for all terms s, if $R_{T_1}(s)$, then $R_{T_2}((\lambda x : T_1.\, t\, (\sigma/\{x\}))\, s)$.

Suppose s is a term and $R_{T_1}(s)$. We must show $R_{T_2}((\lambda x : T_1.\, t\, (\sigma/\{x\}))\, s)$. Since $R_{T_1}(s)$, we have that $\emptyset \vdash s : T_1$ and s converges, so that $s \to^* v$ for some closed value v. By Exercise 4, it follows that $R_{T_1}(v)$. Since $\emptyset \vdash \lambda x : T_1.\, t\, (\sigma/\{x\}) : T_1 \to T_2$ and $\emptyset \vdash s : T_1$, we have that $\emptyset \vdash (\lambda x : T_1.\, t\, (\sigma/\{x\}))\, s : T_2$. By Lemma 5.5, we have that

$$\begin{aligned}(\lambda x : T_1.\, t\, (\sigma/\{x\}))\, s &\to^* (\lambda x : T_1.\, t\, (\sigma/\{x\}))\, v \\ &\to (t\, (\sigma/\{x\}))\, \{(x, v)\} \\ &= t\, \sigma[x \mapsto v].\end{aligned}$$

Thus, by Exercise 4, it will suffice to show that $R_{T_2}(t\, \sigma[x \mapsto v])$. By the inductive hypothesis, $P(\Gamma[x \mapsto T_1], t, T_2)$, it will suffice to show that $\mathsf{dom}(\Gamma[x \mapsto T_1]) \subseteq \mathsf{dom}(\sigma[x \mapsto v])$ and, for all $y \in \mathsf{dom}(\Gamma[x \mapsto T_1])$, $R_{\Gamma[x \mapsto T_1](y)}(\sigma[x \mapsto v](y))$. Since $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(\sigma)$, we have that $\mathsf{dom}(\Gamma[x \mapsto T_1]) = \mathsf{dom}(\Gamma) \cup \{x\} \subseteq \mathsf{dom}(\sigma) \cup \{x\} = \mathsf{dom}(\sigma[x \mapsto v])$. Suppose $y \in \mathsf{dom}(\Gamma[x \mapsto T_1]) = \mathsf{dom}(\Gamma) \cup \{x\}$. We must show that $R_{\Gamma[x \mapsto T_1](y)}(\sigma[x \mapsto v](y))$. There are two cases to consider.

- Suppose $y = x$. Then $R_{T_1}(v)$, $T_1 = \Gamma[x \mapsto T_1](y)$ and $v = \sigma[x \mapsto v](y)$, so that $R_{\Gamma[x \mapsto T_1](y)}(\sigma[x \mapsto v](y))$.
- Suppose $y \neq x$, so that $y \in \mathsf{dom}(\Gamma)$. Thus $R_{\Gamma(y)}(\sigma(y))$, $\Gamma(y) = \Gamma[x \mapsto T_1](y)$ and $\sigma(y) = \sigma[x \mapsto v](y)$, so that $R_{\Gamma[x \mapsto T_1](y)}(\sigma[x \mapsto v](y))$.

(T-App) Suppose Γ is a context, t_1 and t_2 are terms, T_1 and T_2 are types, $\Gamma \vdash t_1 : T_1 \to T_2$ and $\Gamma \vdash t_2 : T_1$, and assume the inductive hypothesis, $P(\Gamma, t_1, T_1 \to T_2)$ and $P(\Gamma, t_2, T_1)$. We must show that $P(\Gamma, t_1\, t_2, T_2)$. Suppose σ is a substitution, $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(\sigma)$, and for all $x \in \mathsf{dom}(\Gamma)$, $R_{\Gamma(x)}(\sigma(x))$. We must show that $R_{T_2}((t_1\, t_2)\, \sigma)$. Since $(t_1\, t_2)\, \sigma = (t_1\, \sigma)(t_2\, \sigma)$, it will suffice to show $R_{T_2}((t_1\, \sigma)(t_2\, \sigma))$. By the inductive hypothesis, we have that $R_{T_1 \to T_2}(t_1\, \sigma)$ and $R_{T_1}(t_2\, \sigma)$. Thus, by the definition of $R_{T_1 \to T_2}(t_1\, \sigma)$, we have $R_{T_2}((t_1\, \sigma)(t_2\, \sigma))$.

Exercise 6

Suppose t is a closed, well-typed term. Thus $\emptyset \vdash t : T$ for some type T. Since $\emptyset \vdash t : T$, $\mathsf{dom}(\emptyset) \subseteq \mathsf{dom}(\emptyset)$, and, for all $x \in \mathsf{dom}(\emptyset)$, $R_{\emptyset(x)}(\emptyset(x))$, Exercise 5 tells us that $R_T(t\, \emptyset)$. Since $\mathsf{FV}(t) \cap \mathsf{dom}(\emptyset) = \emptyset \cap \emptyset = \emptyset$, Exercise 1 tells us that $t\, \emptyset = t$. Thus $R_T(t)$, so that t converges.