2.1 Sets, Relations and Functions

- We write $S \setminus T$ for the *set difference of* S *and* T, i.e., $\{ x \in S \mid x \notin T \}$.
- We write $\mathcal{P}(S)$ for the *powerset of* S, i.e., $\{ X \mid X \subseteq S \}$.
- An n-place *relation* on sets S_1, S_2, ..., S_n is a subset of $S_1 \times S_2 \times \cdots \times S_n$, i.e., a subset of $\{ (x_1, x_2, \ldots, x_n) \mid x_i \in S_i$ for all $1 \leq i \leq n \}$.
- A one-place relation on a set S is called a *predicate* on S. We often write "$P(s)$" instead of "$s \in P$" (literally, this should be "$(s) \in P$").
- A two-place relation on sets S and T is called a *binary relation*. We often write "$s \; R \; t$" instead of $(s, t) \in R$. When $S = T$, R is a binary relation on S.

2.1 Sets, Relations and Functions (Cont.)

- The *domain* ($\mathbf{dom}(R)$) of a relation R on S and T is $\{ s \in S \mid (s, t) \in R$ for some $t \}$, and the *range* ($\mathbf{range}(R)$) of a relation R is $\{ t \in T \mid (s, t) \in R$ for some $s \}$.
- A relation R on sets S and T is called a *partial function* from S to T iff, whenever $(s, t_1), (s, t_2) \in R$, we have $t_1 = t_2$. If, in addition, $\mathbf{dom}(R) = S$, then R is a *total function* from S to T. We write $R(s)$ for the unique t such that $(s, t) \in R$, when such a t exists; otherwise, $R(s)$ is undefined.

2.2: Ordered Sets

A binary relation R on a set S is:

- *reflexive* iff $s\,R\,s$, for all $s \in S$;
- *symmetric* iff $s\,R\,t$ implies $t\,R\,s$, for all $s, t \in S$;
- *transitive* iff $s\,R\,t\,R\,u$ implies $s\,R\,u$, for all $s, t, u \in S$;
- *antisymmetric* iff $s\,R\,t\,R\,s$ implies $s = t$, for all $s, t \in S$.

2.2: Ordered Sets (Cont.)

- A *preorder* R on S is a reflexive and transitive relation.
- If $\leq$ is a preorder, then $s < t$ means $s \leq t$ but $s \neq t$.
- A *partial order* is a preorder that is also antisymmetric.
- A *total order* $\leq$ is a partial order such that, for all $s, t \in S$, either $s \leq t$ or $t \leq s$.
- An *equivalence* on S is a reflexive, transitive and symmetric relation.

2.2 Ordered Sets (Cont.)

- The *transitive closure* of a relation R on S (R^+) is the smallest transitive relation on S that contains R (i.e., the relation R' on S that contains R, is transitive, and such that $R' \subseteq R''$ for all transitive relations R'' on S that contain R).
- How can we construct R^+, thus showing its existence? We will do the construction on the blackboard.
- The *reflexive closure* of R, and the *reflexive and transitive closure* of R (R^*) are defined similarly.

2.3 Sequences

- Sequences are written

$$x_1, x_2, \ldots, x_n$$

  The empty sequence is written $\bullet$ or as a blank (in later chapters, sometimes $\emptyset$ is used).

- $|a|$ is used for the *length* of a. If $a = 3, 1, 5$ and $b = 2, 6, 3, 1$, then: $|a| = 3$ and $|b| = 4$.
- Comma is used for *appending* sequences. If $a = 3, 1, 5$ and $b = 2, 6, 3, 1$, then:

$$a, b = 3, 1, 5, 2, 6, 3, 1$$

$$\bullet, a = a$$

$$a, 2 = 3, 1, 5, 2$$

2.4 Induction

In the following, P is a predicate on the natural numbers.

The *principle of ordinary induction on natural numbers* says that:

> if $P(0)$
> and, for all $i \in \mathbb{N}$, $P(i)$ implies $P(i+1)$,
> then $P(n)$ holds for all $n \in \mathbb{N}$.

The *principle of complete induction on natural numbers* says that:

> if, for each $n \in \mathbb{N}$,
> given $P(i)$ for all $i \in \mathbb{N}$ such that $i < n$,
> we can show $P(n)$,
> then $P(n)$ holds for all $n \in \mathbb{N}$.

Are the induction principles equivalent? Yes, and we will prove this on the blackboard.