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- An n -place *relation* on sets S_1, S_2, \dots, S_n is a subset of $S_1 \times S_2 \times \dots \times S_n$, i.e., a subset of $\{(x_1, x_2, \dots, x_n) \mid x_i \in S_i \text{ for all } 1 \leq i \leq n\}$.

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- A one-place relation on a set S is called a *predicate* on S . We often write “ $P(s)$ ” instead of “ $s \in P$ ” (literally, this should be “ $(s) \in P$ ”).
- A two-place relation on sets S and T is called a *binary relation*. We often write “ $s R t$ ” instead of $(s, t) \in R$. When $S = T$, R is a binary relation on S .

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- The *domain* ($\mathbf{dom}(R)$) of a relation R on S and T is $\{ s \in S \mid (s, t) \in R \text{ for some } t \in T \}$, and the *range* ($\mathbf{range}(R)$) of a relation R is $\{ t \in T \mid (s, t) \in R \text{ for some } s \in S \}$.

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- *antisymmetric* iff $s R t R s$ implies $s = t$, for all $s, t \in S$.

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- An *equivalence* on S is a reflexive, transitive and symmetric relation.

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- How can we construct R^+ , thus showing its existence? We will do the construction on the blackboard.
- The *reflexive closure* of R , and the *reflexive and transitive closure* of R (R^*) are defined similarly.

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- Comma is used for *appending* sequences. If $a = 3, 1, 5$ and $b = 2, 6, 3, 1$, then:

$$a, b = 3, 1, 5, 2, 6, 3, 1$$

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$$a, 2 = 3, 1, 5, 2$$

2.4 Induction

In the following, P is a predicate on the natural numbers.

The *principle of ordinary induction on natural numbers* says that:

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The *principle of complete induction on natural numbers* says that:

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Are the induction principles equivalent? Yes, and we will prove this on the blackboard.