Simply Typed Lambda Calculus: Types

$T ::=$	types:
Bool	type of booleans
$T \to T$	type of functions

Simply Typed Lambda Calculus: Syntax

Syntactic Forms

t ::=	terms:
true	constant true
false	constant false
if t then t else t	conditional
x	variables
$\lambda x \quad . t$	abstraction
$t\, t$	application

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Simply Typed Lambda Calculus: Syntax (Cont.)

Syntactic Forms

$v ::=$	values:
true	true value
false	false value
$\lambda x\!:T.\,t$	abstraction value

Simply Typed Lambda Calculus: Evaluation

Evaluation: $\boxed{t \longrightarrow t'}$

$$\text{if true then } t_2 \text{ else } t_3 \longrightarrow t_2 \qquad \text{(E-IfTrue)}$$

$$\text{if false then } t_2 \text{ else } t_3 \longrightarrow t_3 \qquad \text{(E-IfFalse)}$$

$$\frac{t_1 \longrightarrow t_1'}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \longrightarrow \text{if } t_1' \text{ then } t_2 \text{ else } t_3} \qquad \text{(E-If)}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1\; t_2 \longrightarrow t_1'\; t_2} \qquad \text{(E-App1)}$$

$$\frac{t_2 \longrightarrow t_2'}{v_1\; t_2 \longrightarrow v_1\; t_2'} \qquad \text{(E-App2)}$$

$$(\lambda x{:}\,T_{11}.\,t_{12})v_2 \longrightarrow [x \mapsto v_2]t_{12} \qquad \text{(E-AppAbs)}$$

Simply Typed Lambda Calculus: Typing Contexts

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We abbreviate $\emptyset, x_1 : T_1, \ldots, x_n : T_n$ to $x_1 : T_1, \ldots, x_n : T_n$.

Simply Typed Lambda Calculus: Typing Rules

$\boxed{\Gamma \vdash t : T}$

$$\Gamma \vdash \text{true} : \text{Bool} \qquad \text{(T-True)}$$

$$\Gamma \vdash \text{false} : \text{Bool} \qquad \text{(T-False)}$$

$$\frac{\Gamma \vdash t_1 : \text{Bool} \qquad \Gamma \vdash t_2 : T \qquad \Gamma \vdash t_3 : T}{\Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 : T} \qquad \text{(T-If)}$$

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T} \qquad \text{(T-Var)}$$

$$\frac{\Gamma, x : T_1 \vdash t_2 :}{\Gamma \vdash \lambda x \colon T_1 . t_2 : T_1 \rightarrow} \qquad \text{(T-Abs)}$$

$$\frac{\Gamma \vdash t_1 : T_{11} \rightarrow T_{12} \qquad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1\, t_2 : T_{12}} \qquad \text{(T-App)}$$

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