CIS 570 — Introduction to Formal Language Theory — Fall 2008

Exercise Set 3

Model Answers

Exercise 1

(a) First, we put the following text, which describes M in Forlan's syntax, in the file `es3-ex1-fa`:

```
{states}
A, B, C, D
{start state}
A
{accepting states}
B, C, D
{transitions}
A, % -> B | C | D;
B, 1 -> C; C, 1 -> D;
D, 0 -> C; C, 0 -> B
```

Next, we load M into Forlan, and call it `fa`:

```
- val fa = FA.input "es3-ex1-fa";
val fa = - : fa
```

Next, we declare a function `find` for finding and displaying minimum-length accepting labeled paths in `fa` (M):

```
- fun find s = LP.output("", FA.findAcceptingLP fa (Str.fromString s));
val find = fn : string -> unit
```

Finally, we find and display minimum-length labeled paths explaining why 11001011001, 10011010100 and 00101010110 are accepted by `fa` (M):

```
- find "11001011001";
A, % => B, 1 => C, 1 => D, 0 => C, 0 => B, 1 => C, 0 => B, 1 => C, 1 => D, 0 =>
C, 0 => B, 1 => C
val it = () : unit
- find "10011010100";
A, % => C, 1 => D, 0 => C, 0 => B, 1 => C, 1 => D, 0 => C, 1 => D, 0 => C, 1 =>
D, 0 => C, 0 => B
val it = () : unit
- find "00101010110";
A, % => D, 0 => C, 0 => B, 1 => C, 0 => B, 1 => C, 0 => B, 1 => C, 0 => B, 1 =>
C, 1 => D, 0 => C
val it = () : unit
```

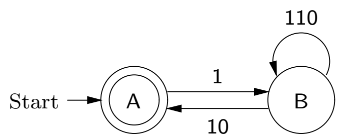
(b) Define a function $\mathbf{diff} \in \{0,1\}^* \to \mathbb{Z}$ by: for all $w \in \{0,1\}^*$,

$$\mathbf{diff}\, w = \text{the number of 1's in } w - \text{the number of 0's in } w.$$

Then $L(M) = \{\, w \in \{0,1\}^* \mid \text{for all substrings } v \text{ of } w,\ -2 \leq \mathbf{diff}\, v \leq 2 \,\}$.

Exercise 2

(a) M is



(b) First, we put the following text, which describes M in Forlan's syntax, in the file `es3-ex2-fa`:

```
{states}
A, B
{start state}
A
{accepting states}
A
{transitions}
A, 1 -> B;
B, 110 -> B;
B, 10 -> A
```

Next, we load M into Forlan, and call it `fa`:

```
- val fa = FA.input "es3-ex2-fa";
val fa = - : fa
```

Next, we declare a function `accepted` for determining whether strings are accepted by `fa` (M):

```
- fun accepted s = FA.accepted fa (Str.fromString s);
val accepted = fn : string -> bool
```

Next, we check that some strings that are in X are accepted by `fa` (M):

```
- map accepted ["%", "110", "111010", "111011010"];
val it = [true,true,true,true] : bool list
```

Finally, we check that some strings that are not in X are not accepted by `fa` (M):

```
- map accepted ["0", "1", "10", "11", "111", "1110", "11100", "11101"];
val it = [false,false,false,false,false,false,false,false] : bool list
```

(c) Define Z_i, for $i \in \{0,1\}$, by: $Z_i = \{\, w \in \{0,1\}^* \mid \mathbf{diff}\, w = i$ and, for all prefixes v of w, $0 \leq \mathbf{diff}\, v \leq 3\,\}$. Hence $Z_0 = X$. By Proposition 3.7.2, we have that $L(M) = \Lambda_{\mathsf{A}}$, because A is M's only accepting state. Thus it will suffice to show that $\Lambda_{\mathsf{A}} = Z_0$.

Lemma ES3.2.1
For all $w \in \{0,1\}^*$*:*

(0) $w \in Z_0$ *iff* $w = \%$ *or* $w = x10$*, for some* $x \in Z_1$*;*

(1) $w \in Z_1$ *iff* $w = x\mathsf{1}$, *for some* $x \in Z_0$, *or* $w = x\mathsf{110}$, *for some* $x \in Z_1$.

Proof. (0, "only if") Suppose $w \in Z_0$. If $w = \%$, then $w = \%$ or $w = x\mathsf{10}$, for some $x \in Z_1$. So, suppose $w \neq \%$. Thus $w = xa$, for some $x \in \{\mathsf{0},\mathsf{1}\}^*$ and $a \in \{\mathsf{0},\mathsf{1}\}$.

Suppose, toward a contradiction, that $a = \mathsf{1}$. Then $\mathbf{diff}\,x+1 = \mathbf{diff}(x\mathsf{1}) = \mathbf{diff}(xa) = \mathbf{diff}\,w = 0$, so that $\mathbf{diff}\,x = -1$. But this is impossible, since x is a prefix of w, and $w \in Z_0$. Thus $a = \mathsf{0}$, so that $w = x\mathsf{0}$.

Since $\mathbf{diff}\,x + -2 = \mathbf{diff}(x\mathsf{0}) = \mathbf{diff}\,w = 0$, we have that $\mathbf{diff}\,x = 2$. Thus $x = yb$, for some $y \in \{\mathsf{0},\mathsf{1}\}^*$ and $b \in \{\mathsf{0},\mathsf{1}\}$. Hence $w = x\mathsf{0} = yb\mathsf{0}$.

Suppose, toward a contradiction, that $b = \mathsf{0}$. Then $\mathbf{diff}\,y + -2 = \mathbf{diff}(y\mathsf{0}) = \mathbf{diff}(yb) = \mathbf{diff}\,x = 2$, so that $\mathbf{diff}\,y = 4$. But this is impossible, since y is a prefix of w, and $w \in Z_0$. Thus $b = \mathsf{1}$, so that $x = y\mathsf{1}$ and $w = y\mathsf{10}$.

Since $\mathbf{diff}\,y + 1 = \mathbf{diff}(y\mathsf{1}) = \mathbf{diff}\,x = 2$, we have that $\mathbf{diff}\,y = 1$. To complete the proof that $y \in Z_1$, suppose v is a prefix of y. Thus v is a prefix of w. But $w \in Z_0$, and thus $0 \leq \mathbf{diff}\,v \leq 3$. Since $w = y\mathsf{10}$ and $y \in Z_1$, we have that $w = \%$ or $w = x\mathsf{10}$, for some $x \in Z_1$.

(0, "if") Suppose $w = \%$ or $w = x\mathsf{10}$, for some $x \in Z_1$. There are two cases to consider.

- Suppose $w = \%$. Then $\mathbf{diff}\,w = \mathbf{diff}\,\% = 0$. To complete the proof that $w \in Z_0$, suppose v is a prefix of w. Then $v = \%$, so that $\mathbf{diff}\,v = 0$, and thus $0 \leq \mathbf{diff}\,v \leq 3$.

- Suppose $w = x\mathsf{10}$, for some $x \in Z_1$. Then $\mathbf{diff}\,w = \mathbf{diff}(x\mathsf{10}) = \mathbf{diff}\,x+1+-2 = 1+1+-2 = 0$. To complete the proof that $w \in Z_0$, suppose v is a prefix of w. If v is a prefix of x, then $0 \leq \mathbf{diff}\,v \leq 3$, since $x \in Z_1$. If $v = x\mathsf{1}$, then $\mathbf{diff}\,v = \mathbf{diff}(x\mathsf{1}) = 1 + 1 = 2$, so that $0 \leq \mathbf{diff}\,v \leq 3$. And, if $v = x\mathsf{10} = w$, then $\mathbf{diff}\,v = \mathbf{diff}\,w = 0$, so that $0 \leq \mathbf{diff}\,v \leq 3$.

(1, "only if") Suppose $w \in Z_1$. Then $\mathbf{diff}\,w = 1$, so that $w \neq \%$. There are two cases to consider.

- Suppose $w = x\mathsf{1}$, for some $x \in \{\mathsf{0},\mathsf{1}\}^*$. Since $\mathbf{diff}\,x + 1 = \mathbf{diff}(x\mathsf{1}) = \mathbf{diff}\,w = 1$, we have that $\mathbf{diff}\,x = 0$. Since x is a prefix of w and $w \in Z_1$, it follows that $x \in Z_0$. Since $w = x\mathsf{1}$ and $x \in Z_0$, we have that $w = x\mathsf{1}$, for some $x \in Z_0$, or $w = x\mathsf{110}$, for some $x \in Z_1$.

- Suppose $w = x\mathsf{0}$, for some $x \in \{\mathsf{0},\mathsf{1}\}^*$. Since $\mathbf{diff}\,x + -2 = \mathbf{diff}(x\mathsf{0}) = \mathbf{diff}\,w = 1$, we have that $\mathbf{diff}\,x = 3$. Thus $x \neq \%$, so that $x = yb$, for some $x \in \{\mathsf{0},\mathsf{1}\}^*$ and $b \in \{\mathsf{0},\mathsf{1}\}$. Hence $w = yb\mathsf{0}$.

  Suppose, toward a contradiction, that $b = \mathsf{0}$. Then $\mathbf{diff}\,y + -2 = \mathbf{diff}(y\mathsf{0}) = \mathbf{diff}(yb) = \mathbf{diff}\,x = 3$, so that $\mathbf{diff}\,y = 5$. But this is impossible, since y is a prefix of w, and $w \in Z_1$. Thus $b = \mathsf{1}$, so that $x = y\mathsf{1}$ and $w = y\mathsf{10}$.

  Since $\mathbf{diff}\,y + 1 = \mathbf{diff}(y\mathsf{1}) = \mathbf{diff}\,x = 3$, we have that $\mathbf{diff}\,y = 2$. Thus $y \neq \%$, so that $y = zc$, for some $z \in \{\mathsf{0},\mathsf{1}\}^*$ and $c \in \{\mathsf{0},\mathsf{1}\}$. Hence $w = zc\mathsf{10}$.

  Suppose, toward a contradiction, that $c = \mathsf{0}$. Then $\mathbf{diff}\,z + -2 = \mathbf{diff}(z\mathsf{0}) = \mathbf{diff}(zc) = \mathbf{diff}\,y = 2$, so that $\mathbf{diff}\,z = 4$. But this is impossible, since z is a prefix of w, and $w \in Z_1$. Thus $c = \mathsf{1}$, so that $y = z\mathsf{1}$ and $w = z\mathsf{110}$.

  Since $\mathbf{diff}\,z + 1 = \mathbf{diff}(z\mathsf{1}) = \mathbf{diff}\,y = 2$, we have that $\mathbf{diff}\,z = 1$. Thus, since z is a prefix of w, and $w \in Z_1$, we have that $z \in Z_1$. Because $w = z\mathsf{110}$ and $z \in Z_1$, we have that $w = x\mathsf{1}$, for some $x \in Z_0$, or $w = x\mathsf{110}$, for some $x \in Z_1$.

(1, “if”) Suppose $w = x\mathsf{1}$, for some $x \in Z_0$, or $w = x\mathsf{110}$, for some $x \in Z_1$. There are two cases to consider.

- Suppose $w = x\mathsf{1}$, for some $x \in Z_0$. Then $\mathbf{diff}\, w = \mathbf{diff}(x\mathsf{1}) = \mathbf{diff}\, x + 1 = 0 + 1 = 1$. To complete the proof that $w \in Z_1$, suppose v is a prefix of w. If v is a prefix of x, then $0 \leq \mathbf{diff}\, v \leq 3$, since $x \in Z_0$. And, if $v = x\mathsf{1} = w$, then $\mathbf{diff}\, v = \mathbf{diff}\, w = 1$, so that $0 \leq \mathbf{diff}\, v \leq 3$.
- Suppose $w = x\mathsf{110}$, for some $x \in Z_1$. Then $\mathbf{diff}\, w = \mathbf{diff}(x\mathsf{110}) = \mathbf{diff}\, x + 1 + 1 + -2 = \mathbf{diff}\, x = 1$. To complete the proof that $w \in Z_1$, suppose v is a prefix of w. If v is a prefix of x, then $0 \leq \mathbf{diff}\, v \leq 3$, since $x \in Z_1$. And, if $v = x\mathsf{1}$, then $\mathbf{diff}\, v = \mathbf{diff}(x\mathsf{1}) = 1 + 1 = 2$, so that $0 \leq \mathbf{diff}(v) \leq 3$. And, if $v = x\mathsf{11}$, then $\mathbf{diff}\, v = \mathbf{diff}(x\mathsf{11}) = 1 + 1 + 1 = 3$, so that $0 \leq \mathbf{diff}\, v \leq 3$. And, if $v = x\mathsf{110} = w$, then $\mathbf{diff}\, v = \mathbf{diff}\, w = 1$, so that $0 \leq \mathbf{diff}\, v \leq 3$.

□

Now we prove a lemma that will tell us that $\Lambda_{\mathsf{A}} \subseteq Z_0$.

Lemma ES3.2.2

(A) For all $w \in \Lambda_{\mathsf{A}}$, $w \in Z_0$.

(B) For all $w \in \Lambda_{\mathsf{B}}$, $w \in Z_1$.

Proof. We proceed by induction on Λ. There are four steps to show.

- (empty string) By Lemma ES3.2.1(0), we have that $\% \in Z_0$.
- $(\mathsf{A}, \mathsf{1} \rightarrow \mathsf{B})$ Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w \in Z_0$. Thus, by Lemma ES3.2.1(1), we have that $w\mathsf{1} \in Z_1$.
- $(\mathsf{B}, \mathsf{10} \rightarrow \mathsf{A})$ Suppose $w \in \Lambda_{\mathsf{B}}$, and assume the inductive hypothesis: $w \in Z_1$. Thus, by Lemma ES3.2.1(0), we have that $w\mathsf{10} \in Z_0$.
- $(\mathsf{B}, \mathsf{110} \rightarrow \mathsf{B})$ Suppose $w \in \Lambda_{\mathsf{B}}$, and assume the inductive hypothesis: $w \in Z_1$. Thus, by Lemma ES3.2.1(1), we have that $w\mathsf{110} \in Z_1$.

□

By Lemma ES3.2.2(A), we have that $\Lambda_{\mathsf{A}} \subseteq Z_0$. Next, we prove a lemma that will tell us that $Z_0 \subseteq \Lambda_{\mathsf{A}}$.

Lemma ES3.2.3

For all $w \in \{\mathsf{0}, \mathsf{1}\}^*$:

(A) if $w \in Z_0$, then $w \in \Lambda_{\mathsf{A}}$;

(B) if $w \in Z_1$, then $w \in \Lambda_{\mathsf{B}}$.

Proof. We proceed by strong string induction. Suppose $w \in \{\mathsf{0}, \mathsf{1}\}^*$, and assume the inductive hypothesis: for all $x \in \{\mathsf{0}, \mathsf{1}\}^*$, if x is a proper substring of w, then:

(A) if $x \in Z_0$, then $x \in \Lambda_{\mathsf{A}}$;

(B) if $x \in Z_1$, then $x \in \Lambda_{\mathsf{B}}$.

We must show that:

(A) if $w \in Z_0$, then $w \in \Lambda_{\mathsf{A}}$;

(B) if $w \in Z_1$, then $w \in \Lambda_{\mathsf{B}}$.

There are two steps to show.

(A) Suppose $w \in Z_0$. We must show that $w \in \Lambda_{\mathsf{A}}$. By Lemma ES3.2.1(0), there are two cases to consider.

- Suppose $w = \%$. By Proposition 3.7.3(1), we have that $w = \% \in \Lambda_{\mathsf{A}}$.
- Suppose $w = x\mathsf{10}$, for some $x \in Z_1$. Since x is a proper substring of w, Part (B) of the inductive hypothesis tells us that $x \in \Lambda_{\mathsf{B}}$. Thus, since $\mathsf{B}, \mathsf{10} \rightarrow \mathsf{A} \in T$, Proposition 3.7.3(2) tells us that $w = x\mathsf{10} \in \Lambda_{\mathsf{A}}$.

(B) Suppose $w \in Z_1$. We must show that $w \in \Lambda_{\mathsf{B}}$. By Lemma ES3.2.1(1), there are two cases to consider.

- Suppose $w = x\mathsf{1}$, for some $x \in Z_0$. Since x is a proper substring of w, Part (A) of the inductive hypothesis tells us that $x \in \Lambda_{\mathsf{A}}$. Thus, since $\mathsf{A}, \mathsf{1} \rightarrow \mathsf{B} \in T$, Proposition 3.7.3(2) tells us that $w = x\mathsf{1} \in \Lambda_{\mathsf{B}}$.
- Suppose $w = x\mathsf{110}$, for some $x \in Z_1$. Since x is a proper substring of w, Part (B) of the inductive hypothesis tells us that $x \in \Lambda_{\mathsf{B}}$. Thus, since $\mathsf{B}, \mathsf{110} \rightarrow \mathsf{B} \in T$, Proposition 3.7.3(2) tells us that $w = x\mathsf{110} \in \Lambda_{\mathsf{B}}$.

□

Because $Z_0 \subseteq \{\mathsf{0}, \mathsf{1}\}^*$, Lemma ES3.2.3(A) tells us that $Z_0 \subseteq \Lambda_{\mathsf{A}}$. Finally, because $\Lambda_{\mathsf{A}} \subseteq Z_0 \subseteq \Lambda_{\mathsf{A}}$, we have that $\Lambda_{\mathsf{A}} = Z_0$.

(d) We say that a string $w \in X$ is *indivisible* iff, for all nonempty, proper prefixes v of w, $\mathbf{diff}\, v > 0$.

Lemma ES3.2.4
For all $n \in \mathbb{N}$, $\mathsf{1}(\mathsf{110})^n\mathsf{10}$ is an indivisible element of X.

Proof. First we use mathematical induction to show that, for all $n \in \mathbb{N}$, $\mathbf{diff}(\mathsf{1}(\mathsf{110})^n) = 1$ and, for all nonempty prefixes v of $\mathsf{1}(\mathsf{110})^n$, $1 \leq \mathbf{diff}\, v \leq 3$.

- (Basis Step) Since $\mathsf{1}(\mathsf{110})^0 = \mathsf{1}$, it will suffice to show that $\mathbf{diff}\, \mathsf{1} = 1$ and that every nonempty prefix of $\mathsf{1}$ has a diff that is between 1 and 3, all of which is obvious.

- (Inductive Step) Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: $\mathbf{diff}(\mathsf{1}(\mathsf{110})^n) = 1$ and, for all nonempty prefixes v of $\mathsf{1}(\mathsf{110})^n$, $1 \leq \mathbf{diff}\, v \leq 3$. We must show that $\mathbf{diff}(\mathsf{1}(\mathsf{110})^{n+1}) = 1$ and, for all nonempty prefixes v of $\mathsf{1}(\mathsf{110})^{n+1}$, $1 \leq \mathbf{diff}\, v \leq 3$. By the inductive hypothesis, we have that $\mathbf{diff}(\mathsf{1}(\mathsf{110})^{n+1}) = \mathbf{diff}(\mathsf{1}(\mathsf{110})^n\mathsf{110}) = \mathbf{diff}(\mathsf{1}(\mathsf{110})^n) +$

$\textbf{diff}(\mathsf{110}) = 1 + 0 = 1$. Finally, suppose v is a nonempty prefix of $\mathsf{1}(\mathsf{110})^{n+1} = \mathsf{1}(\mathsf{110})^n\mathsf{110}$. If v is a nonempty prefix of $\mathsf{1}(\mathsf{110})^n$, then the inductive hypothesis tells us that $1 \leq \textbf{diff}\, v \leq 3$. And, if $v = \mathsf{1}(\mathsf{110})^n\mathsf{1}$, then $\textbf{diff}\, v = 1 + 1 = 2$, so that $1 \leq \textbf{diff}\, v \leq 3$. And, if $v = \mathsf{1}(\mathsf{110})^n\mathsf{11}$, then $\textbf{diff}\, v = 1 + 1 + 1 = 3$, so that $1 \leq \textbf{diff}\, v \leq 3$. And, if $v = \mathsf{1}(\mathsf{110})^n\mathsf{110} = \mathsf{1}(\mathsf{110})^{n+1}$, then $\textbf{diff}\, v = 1$, so that $1 \leq \textbf{diff}\, v \leq 3$.

Now, suppose $n \in \mathbb{N}$. By the result of the above induction, we have that $\textbf{diff}(\mathsf{1}(\mathsf{110})^n\mathsf{10}) = \textbf{diff}(\mathsf{1}(\mathsf{110})^n) + \textbf{diff}(\mathsf{10}) = 1 + -1 = 0$. To finish showing that $\mathsf{1}(\mathsf{110})^n\mathsf{10} \in X$, suppose that v is a prefix of $\mathsf{1}(\mathsf{110})^n\mathsf{10}$. We must show that $0 \leq \textbf{diff}\, v \leq 3$. If $v = \%$, then $\textbf{diff}\, v = 0$, so that $0 \leq \textbf{diff}\, v \leq 3$. And, if v is a nonempty prefix of $\mathsf{1}(\mathsf{110})^n$, then the result of the above induction tells us that $0 \leq 1 \leq \textbf{diff}\, v \leq 3$. And, if $v = \mathsf{1}(\mathsf{110})^n\mathsf{1}$, then $\textbf{diff}\, v = 1 + 1 = 2$, so that $0 \leq \textbf{diff}\, v \leq 3$. Finally, if $v = \mathsf{1}(\mathsf{110})^n\mathsf{10}$, then $\textbf{diff}\, v = 0$, so that $0 \leq \textbf{diff}\, v \leq 3$. This completes the proof that $\mathsf{1}(\mathsf{110})^n\mathsf{10} \in X$.

Finally, we must show that $\mathsf{1}(\mathsf{110})^n\mathsf{10}$ is indivisible. Suppose v is a nonempty, proper prefix of $\mathsf{1}(\mathsf{110})^n\mathsf{10}$. We must show that $\textbf{diff}\, v > 0$. If v is a nonempty prefix of $\mathsf{1}(\mathsf{110})^n$, then the result of the above induction tells us that $1 \leq \textbf{diff}\, v$, so that $\textbf{diff}\, v > 0$. Otherwise, v must be $\mathsf{1}(\mathsf{110})^n\mathsf{1}$, and we have that $\textbf{diff}\, v = 1 + 1 = 2 > 0$. □

Corollary ES3.2.5
X has infinitely many indivisible elements.

Proof. Follows immediately by Lemma ES3.2.4. □

Lemma ES3.2.6
For all finite automata N, if $L(N) = X$, then N has at least two states.

Proof. Suppose N is an FA such that $L(N) = X$. Suppose, toward a contradiction, that N has only one state. Because $L(N) = X$ and X is nonempty, it follows that $s_N \in A_N$.

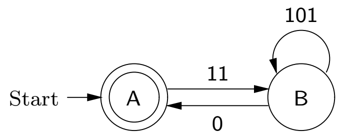
We show that, for all $w \in X$, if w is nonempty and indivisible, then $s_N, w \to s_N \in T_N$. Suppose $w \in X$ is nonempty and indivisible. Because $L(N) = X$, there is a labeled path *lp* such that *lp* is valid for N, and the label of *lp* is w. Of course, all of the states of *lp* are s_N. There are two cases to consider.

- Suppose only one string of *lp* is nonempty. Then one of the steps of *lp* must be labeled by w, so that $s_N, w \to s_N \in T_N$.
- Suppose there are at least two nonempty strings in *lp*. Then *lp* can be divided into a first part, labeled by a nonempty string, x, and a second part, labeled by a nonempty string, y. Thus $w = xy$ and $x, y \in L(N) = X$. Because w is indivisible and x is a nonempty, proper prefix of w, we have that $\textbf{diff}\, x > 0$. But $x \in X$, and thus $\textbf{diff}\, x = 0$—contradiction. Thus $s_N, w \to s_N \in T_N$.

But by Corollary ES3.2.5, there are infinitely many nonempty, indivisible elements of X, and thus it follows that T_N is infinite—contradiction. Thus N has at least two states. □

Because M has two states, Lemma ES3.2.6 tells us that, for all finite automata N, if $L(N) = X$, then N has at least as many states as M.

(e) Let N be the FA



We put the following text, which describes N in Forlan's syntax, in the file `es3-ex2-fa-alt`:

```
{states}
A, B
{start state}
A
{accepting states}
A
{transitions}
A, 11 -> B;
B, 101 -> B;
B, 0 -> A
```

Next, we load N into Forlan, and call it `fa'`:

```
- val fa' = FA.input "es3-ex2-fa-alt";
val fa' = - : fa
```

Finally, we check that `fa` (M) and `fa'` (N) are not isomorphic:

```
- FA.isomorphic(fa, fa');
val it = false : bool
```