CIS 570 — Introduction to Formal Language Theory — Fall 2008

Exercise Set 5

Model Answers

Exercise 1

First, we let the DFA M' be **determSimplify**$(M, \emptyset)$. M' is the same as M, except that the dead state A of M has been renamed to $\langle\mathsf{dead}\rangle$.

Next, we construct the set X of pairs of states of M', as follows.

First, we add to X all pairs consisting of an accepting state and a non-accepting state: $(\mathsf{B}, \langle\mathsf{dead}\rangle)$, $(\langle\mathsf{dead}\rangle, \mathsf{B})$, (B, C), (C, B), $(\mathsf{D}, \langle\mathsf{dead}\rangle)$, $(\langle\mathsf{dead}\rangle, \mathsf{D})$, (D, C) and (C, D). Now, we must handle each of these 8 pairs.

Since there are no transitions leading into B, no pairs can be added using the pairs $(\mathsf{B}, \langle\mathsf{dead}\rangle)$, $(\langle\mathsf{dead}\rangle, \mathsf{B})$, (B, C) and (C, B).

Next, we handle the pairs $(\mathsf{D}, \langle\mathsf{dead}\rangle)$ and $(\langle\mathsf{dead}\rangle, \mathsf{D})$. Since there are no $\mathsf{0}$-transitions leading into D, nothing can be added to X using $(\mathsf{D}, \langle\mathsf{dead}\rangle)$, $(\langle\mathsf{dead}\rangle, \mathsf{D})$ and $\mathsf{0}$-transitions. Since $(\mathsf{C}, \mathsf{1}, \mathsf{D})$, $(\mathsf{B}, \mathsf{1}, \langle\mathsf{dead}\rangle)$, $(\mathsf{D}, \mathsf{1}, \langle\mathsf{dead}\rangle)$ and $(\langle\mathsf{dead}\rangle, \mathsf{1}, \langle\mathsf{dead}\rangle)$, are the $\mathsf{1}$-transitions leading into D and $\langle\mathsf{dead}\rangle$, we add the pairs $(\mathsf{C}, \langle\mathsf{dead}\rangle)$ and $(\langle\mathsf{dead}\rangle, \mathsf{C})$ to X. We would also add the pairs (C, B), (B, C), (C, D) and (D, C) to X, if they were not already there.

Since there are no $\mathsf{0}$-transitions leading into D, nothing can be added to X using (D, C), (C, D) and $\mathsf{0}$-transitions. And, since there are no $\mathsf{1}$-transitions leading into C, nothing can be added to X using (D, C), (C, D) and $\mathsf{1}$-transitions.

We have now handled all of the original elements of X. Next, we must handle the elements of X that we subsequently added: $(\mathsf{C}, \langle\mathsf{dead}\rangle)$ and $(\langle\mathsf{dead}\rangle, \mathsf{C})$. Since $(\mathsf{B}, \mathsf{0}, \mathsf{C})$, $(\mathsf{D}, \mathsf{0}, \mathsf{C})$, $(\mathsf{C}, \mathsf{0}, \langle\mathsf{dead}\rangle)$ and $(\langle\mathsf{dead}\rangle, \mathsf{0}, \langle\mathsf{dead}\rangle)$ are the $\mathsf{0}$-transitions leading into C and $\langle\mathsf{dead}\rangle$, we would add the pairs (B, C), (C, B), $(\mathsf{B}, \langle\mathsf{dead}\rangle)$, $(\langle\mathsf{dead}\rangle, \mathsf{B})$, (D, C), (C, D), $(\mathsf{D}, \langle\mathsf{dead}\rangle)$ and $(\langle\mathsf{dead}\rangle, \mathsf{D})$ to X, if they were not already there. Since there are no $\mathsf{1}$-transitions leading into C, nothing can be added to X using $(\mathsf{C}, \langle\mathsf{dead}\rangle)$, $(\langle\mathsf{dead}\rangle, \mathsf{C})$ and $\mathsf{1}$-transitions.

We have now handled all of the pairs that we have added to X. Thus, X consists of the following 10 pairs: (B, C), $(\mathsf{B}, \langle\mathsf{dead}\rangle)$, (C, B), (C, D), $(\mathsf{C}, \langle\mathsf{dead}\rangle)$, (D, C), $(\mathsf{D}, \langle\mathsf{dead}\rangle)$, $(\langle\mathsf{dead}\rangle, \mathsf{B})$, $(\langle\mathsf{dead}\rangle, \mathsf{C})$ and $(\langle\mathsf{dead}\rangle, \mathsf{D})$.

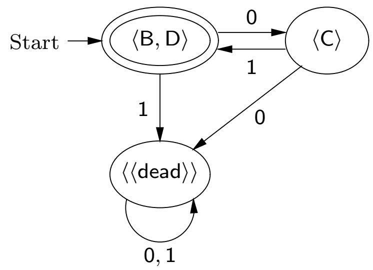
Thus the set Y consists of the following 6 pairs: (B, B), (C, C), (D, D), $(\langle\mathsf{dead}\rangle, \langle\mathsf{dead}\rangle)$, (B, D) and (D, B). Thus the set Z consists of the following equivalence classes: $\{\mathsf{B}, \mathsf{D}\}$, $\{\mathsf{C}\}$ and $\{\langle\mathsf{dead}\rangle\}$. Hence the DFA N has the following states: $\langle\mathsf{B}, \mathsf{D}\rangle$, $\langle\mathsf{C}\rangle$ and $\langle\langle\mathsf{dead}\rangle\rangle$. Since B is the start state of M' and $[\mathsf{B}] = \{\mathsf{B}, \mathsf{D}\}$, we have that the start state of N is $\langle\mathsf{B}, \mathsf{D}\rangle$. Since $\{\mathsf{B}, \mathsf{D}\} \in Z$ and $\mathsf{B} \in A_{M'}$, we have that $\langle\mathsf{B}, \mathsf{D}\rangle \in A_N$. And, since C and $\langle\mathsf{dead}\rangle$ are not accepting states of M', there are no more accepting states of N. It remains to compute the transitions of N.

Since $\{\mathsf{B}, \mathsf{D}\} \in Z$ and $[\delta_{M'}(\mathsf{B}, \mathsf{0})] = [\mathsf{C}] = \{\mathsf{C}\}$, we have that $(\langle\mathsf{B}, \mathsf{D}\rangle, \mathsf{0}, \langle\mathsf{C}\rangle) \in T_N$. Since $\{\mathsf{B}, \mathsf{D}\} \in Z$ and $[\delta_{M'}(\mathsf{B}, \mathsf{1})] = [\langle\mathsf{dead}\rangle] = \{\langle\mathsf{dead}\rangle\}$, we have that $(\langle\mathsf{B}, \mathsf{D}\rangle, \mathsf{1}, \langle\langle\mathsf{dead}\rangle\rangle) \in T_N$.

Since $\{\mathsf{C}\} \in Z$ and $[\delta_{M'}(\mathsf{C}, \mathsf{0})] = [\langle\mathsf{dead}\rangle] = \{\langle\mathsf{dead}\rangle\}$, we have that $(\langle\mathsf{C}\rangle, \mathsf{0}, \langle\langle\mathsf{dead}\rangle\rangle) \in T_N$. Since $\{\mathsf{C}\} \in Z$ and $[\delta_{M'}(\mathsf{C}, \mathsf{1})] = [\mathsf{D}] = \{\mathsf{B}, \mathsf{D}\}$, we have that $(\langle\mathsf{C}\rangle, \mathsf{1}, \langle\mathsf{B}, \mathsf{D}\rangle) \in T_N$.

Since $\{\langle\mathsf{dead}\rangle\} \in Z$ and $[\delta_{M'}(\langle\mathsf{dead}\rangle, 0)] = [\langle\mathsf{dead}\rangle] = \{\langle\mathsf{dead}\rangle\}$, we have that $(\langle\langle\mathsf{dead}\rangle\rangle, 0, \langle\langle\mathsf{dead}\rangle\rangle) \in T_N$. Since $\{\langle\mathsf{dead}\rangle\} \in Z$ and $[\delta_{M'}(\langle\mathsf{dead}\rangle, 1)] = [\langle\mathsf{dead}\rangle] = \{\langle\mathsf{dead}\rangle\}$, we have that $(\langle\langle\mathsf{dead}\rangle\rangle, 1, \langle\langle\mathsf{dead}\rangle\rangle) \in T_N$.

Here is a drawing of N:



To check that our final answer is correct, we put the text

```
{states}
A, B, C, D
{start state}
B
{accepting states}
B, D
{transitions}
A, 0 -> A; A, 1 -> A;
B, 0 -> C; B, 1 -> A;
C, 0 -> A; C, 1 -> D;
D, 0 -> C; D, 1 -> A
```

in the file `es5-ex1-dfa`. Then we invoke Forlan and proceed as follows:

```
- val dfa = DFA.input "es5-ex1-dfa";
val dfa = - : dfa
- val dfa' = DFA.minimize dfa;
val dfa' = - : dfa
- DFA.output("", dfa');
{states} <C>, <B,D>, <<dead>> {start state} <B,D> {accepting states} <B,D>
{transitions}
<C>, 0 -> <<dead>>; <C>, 1 -> <B,D>; <B,D>, 0 -> <C>; <B,D>, 1 -> <<dead>>;
<<dead>>, 0 -> <<dead>>; <<dead>>, 1 -> <<dead>>
val it = () : unit
```

It is easy to check that the outputted DFA is N.

Exercise 2

Let the DFAs M_1 and M_2 be:

(M_1) (M_2)

Then

$$
\begin{aligned}
L(N_1) &= L(\mathbf{minus}(M_1, M_2)) = L(M_1) - L(M_2) = \{0\}^* - \{1\}^* = \{0\}\{0\}^*, \\
L(N_2) &= L(\mathbf{inter}(M_1, \mathbf{complement}(M_2, \emptyset))) = L(M_1) \cap L(\mathbf{complement}(M_2, \emptyset)) \\
&= L(M_1) \cap ((\mathbf{alphabet}(L(M_2)) \cup \emptyset)^* - L(M_2)) = \{0\}^* \cap (\{1\}^* - \{1\}^*) \\
&= \{0\}^* \cap \emptyset = \emptyset.
\end{aligned}
$$

Thus N_1 and N_2 are not equivalent.

We can use Forlan to check that our answer is correct, as follows:

```
- val dfa1 = DFA.input "";
@ {states} A {start state} A {accepting states} A {transitions} A, 0 -> A
@ .
val dfa1 = - : dfa
- val dfa2 = DFA.input "";
@ {states} A {start state} A {accepting states} A {transitions} A, 1 -> A
@ .
val dfa2 = - : dfa
- val dfa'1 = DFA.minus(dfa1, dfa2);
val dfa'1 = - : dfa
- val dfa'2 = DFA.inter(dfa1, DFA.complement(dfa2, SymSet.fromString ""));
val dfa'2 = - : dfa
- DFA.relationship(dfa'1, dfa'2);
first language is a proper superset of second language : "0" is in first
language but is not in second language
val it = () : unit
```

Exercise 3

(a) Define functions **HasSub** $\in \{0,1\}^* \rightarrow \mathbf{Lan}$, **HasNotSub** $\in \{0,1\}^* \rightarrow \mathbf{Lan}$, **HasSubs** $\in \{0,1\}^* \times \{0,1\}^* \rightarrow \mathbf{Lan}$ and **HasNotSubs** $\in \{0,1\}^* \times \{0,1\}^* \rightarrow \mathbf{Lan}$ by:

- for all $x \in \{0,1\}^*$, $\mathbf{HasSub}\, x = \{\, w \in \{0,1\}^* \mid x \text{ is a substring of } w \,\}$;
- for all $x \in \{0,1\}^*$, $\mathbf{HasNotSub}\, x = \{\, w \in \{0,1\}^* \mid x \text{ is not a substring of } w \,\}$;
- for all $x, y \in \{0,1\}^*$,

$$\mathbf{HasSubs}(x, y) = \{\, w \in \{0,1\}^* \mid x \text{ is a substring of } w \text{ and } y \text{ is a substring of } w \,\};$$

- for all $x, y \in \{0,1\}^*$,

$$\mathbf{HasNotSubs}(x, y) = \{\, w \in \{0,1\}^* \mid x \text{ is not a substring of } w \text{ and } y \text{ is not a substring of } w \,\}.$$

Lemma ES5.3.1

(1) For all $x \in \{0,1\}^*$, $\mathbf{HasSub}\, x = \{0,1\}^*\,\{x\}\,\{0,1\}^*$.

(2) For all $x \in \{0,1\}^*$, $\mathbf{HasNotSub}\, x = \{0,1\}^* - \mathbf{HasSub}\, x$.

(3) For all $x, y \in \{0,1\}^*$, $\mathbf{HasSubs}(x, y) = \mathbf{HasSub}\, x \cap \mathbf{HasSub}\, y$.

(4) For all $x, y \in \{0,1\}^*$, $\mathbf{HasNotSubs}(x, y) = \mathbf{HasNotSub}\, x \cap \mathbf{HasNotSub}\, y$.

(5) For all $x, y \in \{0,1\}^*$, $\mathbf{IFF}(x, y) = \mathbf{HasSubs}(x, y) \cup \mathbf{HasNotSubs}(x, y)$.

Proof.

(1) Suppose $x \in \{0,1\}^*$. We must show that $\mathbf{HasSub}\, x = \{0,1\}^*\,\{x\}\,\{0,1\}^*$. It will suffice to show that $\mathbf{HasSub}\, x \subseteq \{0,1\}^*\,\{x\}\,\{0,1\}^* \subseteq \mathbf{HasSub}\, x$.

- Suppose $w \in \mathbf{HasSub}\, x$. Then $w \in \{0,1\}^*$ and x is a substring of w. Thus $w = uxv$ for some $u, v \in \{0,1\}^*$, so that $w = uxv \in \{0,1\}^*\,\{x\}\,\{0,1\}^*$.
- Suppose $w \in \{0,1\}^*\,\{x\}\,\{0,1\}^*$. Then $w = uxv$ for some $u, v \in \{0,1\}^*$. Hence $w \in \{0,1\}^*$ and x is a substring of w, so that $w \in \mathbf{HasSub}\, x$.

(2) Suppose $x \in \{0,1\}^*$. We must show that $\mathbf{HasNotSub}\, x = \{0,1\}^* - \mathbf{HasSub}\, x$. It will suffice to show that $\mathbf{HasNotSub}\, x \subseteq \{0,1\}^* - \mathbf{HasSub}\, x \subseteq \mathbf{HasNotSub}\, x$.

- Suppose $w \in \mathbf{HasNotSub}\, x$. Then $w \in \{0,1\}^*$ and x is not a substring of w. Suppose, toward a contradiction, that $w \in \mathbf{HasSub}\, x$. Then x is a substring of w—contradiction. Thus $w \notin \mathbf{HasSub}\, x$, completing the proof that $w \in \{0,1\}^* - \mathbf{HasSub}\, x$.
- Suppose $w \in \{0,1\}^* - \mathbf{HasSub}\, x$. Then $w \in \{0,1\}^*$ and $w \notin \mathbf{HasSub}\, x$. Suppose, toward a contradiction, that x is a substring of w. Then $w \in \mathbf{HasSub}\, x$—contradiction. Hence x is not a substring of w, completing the proof that $w \in \mathbf{HasNotSub}\, x$.

(3) Suppose $x, y \in \{0,1\}^*$. We must show that $\mathbf{HasSubs}(x, y) = \mathbf{HasSub}\, x \cap \mathbf{HasSub}\, y$. It will suffice to show that $\mathbf{HasSubs}(x, y) \subseteq \mathbf{HasSub}\, x \cap \mathbf{HasSub}\, y \subseteq \mathbf{HasSubs}(x, y)$.

- Suppose $w \in \mathbf{HasSubs}(x, y)$. Then $w \in \{0,1\}^*$, x is a substring of w, and y is a substring of w. Thus $w \in \mathbf{HasSub}\, x$ and $w \in \mathbf{HasSub}\, y$, so that $w \in \mathbf{HasSub}\, x \cap \mathbf{HasSub}\, y$.
- Suppose $w \in \mathbf{HasSub}\, x \cap \mathbf{HasSub}\, y$. Thus $w \in \mathbf{HasSub}\, x$ and $w \in \mathbf{HasSub}\, y$, so that $w \in \{0,1\}^*$, x is a substring of w, and y is a substring of w. Thus $w \in \mathbf{HasSubs}(x, y)$.

(4) Suppose $x, y \in \{0,1\}^*$. We must show that $\mathbf{HasNotSubs}(x, y) = \mathbf{HasNotSub}\, x \cap \mathbf{HasNotSub}\, y$. It will suffice to show that $\mathbf{HasNotSubs}(x, y) \subseteq \mathbf{HasNotSub}\, x \cap \mathbf{HasNotSub}\, y \subseteq \mathbf{HasNotSubs}(x, y)$.

- Suppose $w \in \mathbf{HasNotSubs}(x, y)$. Then $w \in \{0,1\}^*$, x is not a substring of w, and y is not a substring of w. Thus $w \in \mathbf{HasNotSub}\, x$ and $w \in \mathbf{HasNotSub}\, y$, so that $w \in \mathbf{HasNotSub}\, x \cap \mathbf{HasNotSub}\, y$.
- Suppose $w \in \mathbf{HasNotSub}\, x \cap \mathbf{HasNotSub}\, y$. Thus $w \in \mathbf{HasNotSub}\, x$ and $w \in \mathbf{HasNotSub}\, y$, so that $w \in \{0,1\}^*$, x is not a substring of w, and y is not a substring of w. Thus $w \in \mathbf{HasNotSubs}(x, y)$.

(5) Suppose $x, y \in \{\mathsf{0}, \mathsf{1}\}^*$. We must show that $\mathbf{IFF}(x, y) = \mathbf{HasSubs}(x, y) \cup \mathbf{HasNotSubs}(x, y)$. It will suffice to show that $\mathbf{IFF}(x, y) \subseteq \mathbf{HasSubs}(x, y) \cup \mathbf{HasNotSubs}(x, y) \subseteq \mathbf{IFF}(x, y)$.

- Suppose $w \in \mathbf{IFF}(x, y)$. Then $w \in \{\mathsf{0}, \mathsf{1}\}^*$ and x is a substring of w iff y is a substring of w. There are two cases to consider.
  - Suppose x is a substring of w. Then y is a substring of w. Hence $w \in \mathbf{HasSubs}(x, y) \subseteq \mathbf{HasSubs}(x, y) \cup \mathbf{HasNotSubs}(x, y)$.
  - Suppose x is not a substring of w. Thus y is not a substring of w. Hence $w \in \mathbf{HasNotSubs}(x, y) \subseteq \mathbf{HasSubs}(x, y) \cup \mathbf{HasNotSubs}(x, y)$.
- Suppose $w \in \mathbf{HasSubs}(x, y) \cup \mathbf{HasNotSubs}(x, y)$. There are two cases to consider.
  - Suppose $w \in \mathbf{HasSubs}(x, y)$. Then $w \in \{\mathsf{0}, \mathsf{1}\}^*$, x is a substring of w, and y is a substring of w. Thus x is a substring of w iff y is a substring of w, so that $w \in \mathbf{IFF}(x, y)$.
  - Suppose $w \in \mathbf{HasNotSubs}(x, y)$. Then $w \in \{\mathsf{0}, \mathsf{1}\}^*$, x is not a substring of w, and y is not a substring of w. Thus x is a substring of w iff y is a substring of w, so that $w \in \mathbf{IFF}(x, y)$.

□

Next, we define some useful functions. Define $\mathbf{efaToDFA} \in \mathbf{EFA} \to \mathbf{DFA}$ by: $\mathbf{efaToDFA} = \mathbf{nfaToDFA} \circ \mathbf{efaToNFA}$. Then we have that, for all $M \in \mathbf{EFA}$,

$$L(\mathbf{efaToDFA}\, M) = L(\mathbf{nfaToDFA}(\mathbf{efaToNFA}\, M)) = L(\mathbf{efaToNFA}\, M) = L(M).$$

Define $\mathbf{regToEFA} \in \mathbf{Reg} \to \mathbf{EFA}$ by: $\mathbf{regToEFA} = \mathbf{faToEFA} \circ \mathbf{regToFA}$. Then we have that, for all $\alpha \in \mathbf{Reg}$,

$$L(\mathbf{regToEFA}\, \alpha) = L(\mathbf{faToEFA}(\mathbf{regToFA}\, \alpha)) = L(\mathbf{regToFA}\, \alpha) = L(\alpha).$$

Define $\mathbf{regToDFA} \in \mathbf{Reg} \to \mathbf{DFA}$ by: $\mathbf{regToDFA} = \mathbf{efaToDFA} \circ \mathbf{regToEFA}$. Then we have that, for all $\alpha \in \mathbf{Reg}$,

$$L(\mathbf{regToDFA}\, \alpha) = L(\mathbf{efaToDFA}(\mathbf{regToEFA}\, \alpha)) = L(\mathbf{regToEFA}\, \alpha) = L(\alpha).$$

Define $\mathbf{faToDFA} \in \mathbf{FA} \to \mathbf{DFA}$ by: $\mathbf{faToDFA} = \mathbf{efaToDFA} \circ \mathbf{faToEFA}$. Then we have that, for all $M \in \mathbf{FA}$,

$$L(\mathbf{faToDFA}\, M) = L(\mathbf{efaToDFA}(\mathbf{faToEFA}\, M)) = L(\mathbf{faToEFA}\, M) = L(M).$$

Define $\mathbf{minAndRen} \in \mathbf{DFA} \to \mathbf{DFA}$ by: for all $M \in \mathbf{DFA}$,

$$\mathbf{minAndRen}\, M = \mathbf{renameStatesCanonically}(\mathbf{minimize}\, M).$$

Then, for all $M \in \mathbf{DFA}$,

$$\begin{aligned} L(\mathbf{minAndRen}\, M) &= L(\mathbf{renameStatesCanonically}(\mathbf{minimize}\, M)) \\ &= L(\mathbf{minimize}\, M) = L(M), \end{aligned}$$

and $\mathbf{minAndRen}\, M$ has no more states than any DFA that is equivalent to M.

Define the DFA **allStrDFA** by:

$$\mathbf{allStrDFA} = \mathbf{minAndRen}(\mathbf{regToDFA}((\mathsf{0} + \mathsf{1})^*)).$$

Then, we have that

$$\begin{aligned}
L(\mathbf{allStrDFA}) &= L(\mathbf{minAndRen}(\mathbf{regToDFA}((\mathsf{0} + \mathsf{1})^*))) \\
&= L(\mathbf{regToDFA}((\mathsf{0} + \mathsf{1})^*)) \\
&= L((\mathsf{0} + \mathsf{1})^*) = \{\mathsf{0}, \mathsf{1}\}^*.
\end{aligned}$$

Let the FA **allStrFA** be **allStrDFA**. Thus $L(\mathbf{allStrFA}) = L(\mathbf{allStrDFA}) = \{\mathsf{0}, \mathsf{1}\}^*$.

Define $\mathbf{hasSubFA} \in \{\mathsf{0}, \mathsf{1}\}^* \to \mathbf{FA}$ by: for all $x \in \{\mathsf{0}, \mathsf{1}\}^*$,

$$\mathbf{hasSubFA}\, x = \mathbf{concat}(\mathbf{allStrFA}, \mathbf{concat}(\mathbf{strToFA}\, x, \mathbf{allStrFA})).$$

Then, we have that, for all $x \in \{\mathsf{0}, \mathsf{1}\}^*$,

$$\begin{aligned}
L(\mathbf{hasSubFA}\, x) &= L(\mathbf{concat}(\mathbf{allStrFA}, \mathbf{concat}(\mathbf{strToFA}\, x, \mathbf{allStrFA}))) \\
&= L(\mathbf{allStrFA})\, L(\mathbf{strToFA}\, x)\, L(\mathbf{allStrFA}) \\
&= \{\mathsf{0}, \mathsf{1}\}^* \{x\}\, \{\mathsf{0}, \mathsf{1}\}^* \\
&= \mathbf{HasSub}\, x,
\end{aligned}$$

by Lemma ES5.3.1(1).

Define $\mathbf{hasSubDFA} \in \{\mathsf{0}, \mathsf{1}\}^* \to \mathbf{DFA}$ by:

$$\mathbf{hasSubDFA} = \mathbf{minAndRen} \circ \mathbf{faToDFA} \circ \mathbf{hasSubFA}.$$

Then, we have that, for all $x \in \{\mathsf{0}, \mathsf{1}\}^*$,

$$\begin{aligned}
L(\mathbf{hasSubDFA}\, x) &= L(\mathbf{minAndRen}(\mathbf{faToDFA}(\mathbf{hasSubFA}\, x))) \\
&= L(\mathbf{faToDFA}(\mathbf{hasSubFA}\, x)) \\
&= L(\mathbf{hasSubFA}\, x) \\
&= \mathbf{HasSub}\, x.
\end{aligned}$$

Define $\mathbf{hasSubEFA} \in \{\mathsf{0}, \mathsf{1}\}^* \to \mathbf{EFA}$ by: $\mathbf{hasSubEFA} = \mathbf{hasSubDFA}$. Then, for all $x \in \{\mathsf{0}, \mathsf{1}\}^*$,

$$L(\mathbf{hasSubEFA}\, x) = L(\mathbf{hasSubDFA}\, x) = \mathbf{HasSub}\, x.$$

Define $\mathbf{hasNotSubDFA} \in \{\mathsf{0}, \mathsf{1}\}^* \to \mathbf{DFA}$ by: for all $x \in \{\mathsf{0}, \mathsf{1}\}^*$,

$$\mathbf{hasNotSubDFA}\, x = \mathbf{minus}(\mathbf{allStrDFA}, \mathbf{hasSubDFA}\, x).$$

Then, we have that, for all $x \in \{\mathsf{0}, \mathsf{1}\}^*$,

$$\begin{aligned}
L(\mathbf{hasNotSubDFA}\, x) &= L(\mathbf{minus}(\mathbf{allStrDFA}, \mathbf{hasSubDFA}\, x)) \\
&= L(\mathbf{allStrDFA}) - L(\mathbf{hasSubDFA}\, x) \\
&= \{\mathsf{0}, \mathsf{1}\}^* - \mathbf{HasSub}\, x \\
&= \mathbf{HasNotSub}\, x,
\end{aligned}$$

by Lemma ES5.3.1(2).

Define $\mathbf{hasNotSubEFA} \in \{0,1\}^* \to \mathbf{EFA}$ by: $\mathbf{hasNotSubEFA} = \mathbf{hasNotSubDFA}$. Then, for all $x \in \{0,1\}^*$,

$$L(\mathbf{hasNotSubEFA}\, x) = L(\mathbf{hasNotSubDFA}\, x) = \mathbf{HasNotSub}\, x.$$

Define $\mathbf{hasSubsEFA} \in \{0,1\}^* \times \{0,1\}^* \to \mathbf{EFA}$ by: for all $x, y \in \{0,1\}^*$,

$$\mathbf{hasSubsEFA}(x,y) = \mathbf{inter}(\mathbf{hasSubEFA}\, x, \mathbf{hasSubEFA}\, y).$$

Then, for all $x, y \in \{0,1\}^*$,

$$\begin{aligned} L(\mathbf{hasSubsEFA}(x,y)) &= L(\mathbf{inter}(\mathbf{hasSubEFA}\, x, \mathbf{hasSubEFA}\, y)) \\ &= L(\mathbf{hasSubEFA}\, x) \cap L(\mathbf{hasSubEFA}\, y) \\ &= \mathbf{HasSub}\, x \cap \mathbf{HasSub}\, y \\ &= \mathbf{HasSubs}(x,y), \end{aligned}$$

by Lemma ES5.3.1(3).

Define $\mathbf{hasNotSubsEFA} \in \{0,1\}^* \times \{0,1\}^* \to \mathbf{EFA}$ by: for all $x, y \in \{0,1\}^*$,

$$\mathbf{hasNotSubsEFA}(x,y) = \mathbf{inter}(\mathbf{hasNotSubEFA}\, x, \mathbf{hasNotSubEFA}\, y).$$

Then, for all $x, y \in \{0,1\}^*$,

$$\begin{aligned} L(\mathbf{hasNotSubsEFA}(x,y)) &= L(\mathbf{inter}(\mathbf{hasNotSubEFA}\, x, \mathbf{hasNotSubEFA}\, y)) \\ &= L(\mathbf{hasNotSubEFA}\, x) \cap L(\mathbf{hasNotSubEFA}\, y) \\ &= \mathbf{HasNotSub}\, x \cap \mathbf{HasNotSub}\, y \\ &= \mathbf{HasNotSubs}(x,y), \end{aligned}$$

by Lemma ES5.3.1(4).

Define $\mathbf{iffEFA} \in \{0,1\}^* \times \{0,1\}^* \to \mathbf{EFA}$ by: for all $x, y \in \{0,1\}^*$,

$$\mathbf{iffEFA}(x,y) = \mathbf{union}(\mathbf{hasSubsEFA}(x,y), \mathbf{hasNotSubsEFA}(x,y)).$$

Then, for all $x \in \{0,1\}^*$,

$$\begin{aligned} L(\mathbf{iffEFA}(x,y)) &= L(\mathbf{union}(\mathbf{hasSubsEFA}(x,y), \mathbf{hasNotSubsEFA}(x,y))) \\ &= L(\mathbf{hasSubsEFA}(x,y)) \cup L(\mathbf{hasNotSubsEFA}(x,y)) \\ &= \mathbf{HasSubs}(x,y) \cup \mathbf{HasNotSubs}(x,y) \\ &= \mathbf{IFF}(x,y), \end{aligned}$$

by Lemma ES5.3.1(5).

Finally, define $\mathbf{iffDFA} \in \{0,1\}^* \times \{0,1\}^* \to \mathbf{DFA}$ by:

$$\mathbf{iffDFA} = \mathbf{minAndRen} \circ \mathbf{efaToDFA} \circ \mathbf{iffEFA}.$$

Then, we have that, for all $x, y \in \{0, 1\}^*$,

$$\begin{aligned}
L(\textbf{iffDFA}(x, y)) &= L(\textbf{minAndRen}(\textbf{efaToDFA}(\textbf{iffEFA}(x, y)))) \\
&= L(\textbf{efaToDFA}(\textbf{iffEFA}(x, y))) \\
&= L(\textbf{iffEFA}(x, y)) \\
&= \textbf{IFF}(x, y),
\end{aligned}$$

and $\textbf{iffDFA}(x, y)$ has as few states as possible, since its last step is **minAndRen**.

(b) First, we put the text

```
val efaToDFA  = nfaToDFA o efaToNFA;
val regToEFA  = faToEFA o regToFA;
val regToDFA  = efaToDFA o regToEFA;
val faToDFA   = efaToDFA o faToEFA;
val minAndRen = DFA.renameStatesCanonically o DFA.minimize;

val allStrDFA = minAndRen(regToDFA(Reg.fromString "(0 + 1)*"));
val allStrFA  = injDFAToFA allStrDFA;

fun hasSubFA x = FA.concat(allStrFA, FA.concat(strToFA x, allStrFA));

val hasSubDFA = minAndRen o faToDFA o hasSubFA;
val hasSubEFA = injDFAToEFA o hasSubDFA;

fun hasNotSubDFA x = DFA.minus(allStrDFA, hasSubDFA x);

val hasNotSubEFA = injDFAToEFA o hasNotSubDFA;

fun hasSubsEFA(x, y) = EFA.inter(hasSubEFA x, hasSubEFA y);

fun hasNotSubsEFA(x, y) = EFA.inter(hasNotSubEFA x, hasNotSubEFA y);

fun iffEFA(x, y) = EFA.union(hasSubsEFA(x, y), hasNotSubsEFA(x, y));

val iffDFA = minAndRen o efaToDFA o iffEFA;
```

in the file `iff.sml`. Then we invoke Forlan and proceed as follows:

```
- use "iff.sml";
[opening iff.sml]
val efaToDFA = fn : efa -> dfa
val regToEFA = fn : reg -> efa
val regToDFA = fn : reg -> dfa
val faToDFA = fn : fa -> dfa
val minAndRen = fn : dfa -> dfa
val allStrDFA = - : dfa
val allStrFA = - : fa
```

```
val hasSubFA = fn : str -> fa
val hasSubDFA = fn : str -> dfa
val hasSubEFA = fn : str -> efa
val hasNotSubDFA = fn : str -> dfa
val hasNotSubEFA = fn : str -> efa
val hasSubsEFA = fn : str * str -> efa
val hasNotSubsEFA = fn : str * str -> efa
val iffEFA = fn : str * str -> efa
val iffDFA = fn : str * str -> dfa
val it = () : unit
- val dfa = iffDFA(Str.fromString "0011", Str.fromString "1100");
val dfa = - : dfa
- DFA.output("", dfa);
{states} A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P {start state} A
{accepting states} A, B, C, E, F, H, J, P
{transitions}
A, 0 -> E; A, 1 -> B; B, 0 -> E; B, 1 -> C; C, 0 -> F; C, 1 -> C; D, 0 -> G;
D, 1 -> D; E, 0 -> J; E, 1 -> B; F, 0 -> K; F, 1 -> B; G, 0 -> K; G, 1 -> D;
H, 0 -> E; H, 1 -> O; I, 0 -> G; I, 1 -> P; J, 0 -> J; J, 1 -> H; K, 0 -> K;
K, 1 -> I; L, 0 -> L; L, 1 -> M; M, 0 -> L; M, 1 -> O; N, 0 -> P; N, 1 -> M;
O, 0 -> N; O, 1 -> O; P, 0 -> P; P, 1 -> P
val it = () : unit
```

Here is a drawing of `dfa`: