CIS 570 — Introduction to Formal Language Theory — Fall 2007

Exercise Set 1

Model Answers

Exercise 1

(Basis Step) We have that

$$\begin{aligned} f(0) &= 0 \qquad \text{(definition of } f(0)) \\ &= \frac{0^2 - 0}{2}. \end{aligned}$$

(Inductive Step) Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis:

$$f(n) = \frac{n^2 - n}{2}.$$

We must show that

$$f(n+1) = \frac{(n+1)^2 - (n+1)}{2}.$$

Then,

$$\begin{aligned} f(n+1) &= f(n) + n && \text{(definition of } f(n+1)) \\ &= \frac{n^2 - n}{2} + n && \text{(inductive hypothesis)} \\ &= \frac{n^2 - n}{2} + \frac{2n}{2} \\ &= \frac{n^2 - n + 2n}{2} \\ &= \frac{n^2 + 2n + 1 - n - 1}{2} \\ &= \frac{(n+1)^2 - (n+1)}{2}. \end{aligned}$$

Exercise 2

First, we prove some useful lemmas.

Lemma ES1.2.1
$c(a - b) = 1$.

Proof. We have that

$$c(a-b) = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}) = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2} + \frac{\sqrt{5}-1}{2}) = \frac{1}{\sqrt{5}} \cdot \frac{2\sqrt{5}}{2} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1.$$

□

Lemma ES1.2.2

(1) $aa = a + 1$.

(2) $bb = b + 1$.

Proof.

(1) We have that

$$\begin{aligned} aa &= \frac{1+\sqrt{5}}{2} \cdot \frac{1+\sqrt{5}}{2} = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2} + \frac{2}{2} \\ &= \frac{1+\sqrt{5}}{2} + 1 = a + 1. \end{aligned}$$

(2) We have that

$$\begin{aligned} bb &= \frac{1-\sqrt{5}}{2} \cdot \frac{1-\sqrt{5}}{2} = \frac{1+-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} + \frac{2}{2} \\ &= \frac{1-\sqrt{5}}{2} + 1 = b + 1. \end{aligned}$$

□

Lemma ES1.2.3

(1) For all $n \in \mathbb{N} - \{0, 1\}$, $a^n = a^{n-1} + a^{n-2}$.

(2) For all $n \in \mathbb{N} - \{0, 1\}$, $b^n = b^{n-1} + b^{n-2}$.

Proof.

(1) Suppose $n \in \mathbb{N} - \{0, 1\}$. We have that

$$\begin{aligned} a^n &= (aa)a^{n-2} \\ &= (a+1)a^{n-2} && \text{(Lemma ES1.2.2(1))} \\ &= aa^{n-2} + 1a^{n-2} \\ &= a^{n-1} + a^{n-2}. \end{aligned}$$

(2) Suppose $n \in \mathbb{N} - \{0, 1\}$. We have that

$$\begin{aligned} b^n &= (bb)b^{n-2} \\ &= (b+1)b^{n-2} && \text{(Lemma ES1.2.2(2))} \\ &= bb^{n-2} + 1b^{n-2} \\ &= b^{n-1} + b^{n-2}. \end{aligned}$$

□

Now, we use our lemmas to prove that, for all $n \in \mathbb{N}$, $f(n) = c(a^n - b^n)$. We proceed by strong induction. Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: for all $m \in \mathbb{N}$, if $m < n$, then $f(m) = c(a^m - b^m)$. We must show that $f(n) = c(a^n - b^n)$. There are three cases to consider.

- $(n = 0)$ We have that

$$f(n) = 0 = c \cdot 0 = c(1-1) = c(a^0 - b^0) = c(a^n - b^n).$$

- $(n = 1)$ We have that

$$\begin{aligned} f(n) &= 1 \\ &= c(a-b) && \text{(Lemma ES1.2.1)} \\ &= c(a^1 - b^1) = c(a^n - b^n). \end{aligned}$$

- $(n \geq 2)$ We have that

$$\begin{aligned} f(n) &= f(n-1) + f(n-2) \\ &= c(a^{n-1} - b^{n-1}) + c(a^{n-2} - b^{n-2}) && \text{(inductive hypothesis on } n-1, n-2 < n) \\ &= c((a^{n-1} - b^{n-1}) + (a^{n-2} - b^{n-2})) \\ &= c((a^{n-1} + a^{n-2}) - (b^{n-1} + b^{n-2})) \\ &= c(a^n - b^n) && \text{(Lemma ES1.2.3).} \end{aligned}$$

Exercise 3

(a) Suppose A and B are sets. We must show that

$$A = (A \cap B) \cup (A - B).$$

It will suffice to show that

$$A \subseteq (A \cap B) \cup (A - B) \subseteq A.$$

$(A \subseteq (A \cap B) \cup (A - B))$ Suppose $w \in A$. We must show that $w \in (A \cap B) \cup (A - B)$. There are two cases to consider.

- Suppose $w \in B$. Then $w \in A$ and $w \in B$, so that $w \in A \cap B \subseteq (A \cap B) \cup (A - B)$.
- Suppose $w \notin B$. Then $w \in A$ and $w \notin B$, so that $w \in A - B \subseteq (A \cap B) \cup (A - B)$.

$((A \cap B) \cup (A - B) \subseteq A)$ Suppose $w \in (A \cap B) \cup (A - B)$. We must show that $w \in A$. There are two cases to consider.

- Suppose $w \in A \cap B$. Thus $w \in A$ and $w \in B$, so that $w \in A$.
- Suppose $w \in A - B$. Thus $w \in A$ and $w \notin B$, so that $w \in A$.

(b) Suppose, toward a contradiction, that, for all sets A, B and C, $(A-B)-C = A-(B-C)$. Let

$$A = \{0, 1\} \qquad \text{and} \qquad B = \{0\} \qquad \text{and} \qquad C = \{1\}.$$

Thus $(A - B) - C = A - (B - C)$, so that

$$\begin{aligned} \emptyset &= \{1\} - \{1\} = (\{0,1\} - \{0\}) - \{1\} = (A - B) - C \\ &= A - (B - C) = \{0,1\} - (\{0\} - \{1\}) = \{0,1\} - \{0\} = \{1\}. \end{aligned}$$

Hence $\emptyset = \{1\}$—contradiction. Thus it is not the case that, for all sets A, B and C, $(A - B) - C = A - (B - C)$.

Exercise 4

(a) We use induction on X to show that, for all $w \in X$, $w \in Y$. There are three steps to show.

(1) We must show that $1 \in Y$. Clearly $1 \in \{0,1\}^*$, and $\mathbf{diff}(1) = 1$. So, it remains to show that, for all prefixes v of 1, $\mathbf{diff}(v) \le 1$. Suppose v is a prefix of 1. We must show that $\mathbf{diff}(v) \le 1$. There are two cases to consider.

- Suppose $v = \%$. Then $\mathbf{diff}(v) = \mathbf{diff}(\%) = 0 \le 1$.
- Suppose $v = 1$. Then $\mathbf{diff}(v) = \mathbf{diff}(1) = 1 \le 1$.

(2) Suppose $x, y \in X$ and assume the inductive hypothesis: $x, y \in Y$. We must show that $x0y \in Y$. Because $x, y \in Y \subseteq \{0,1\}^*$ and $0 \in \{0,1\}^*$, we have that $x0y \in \{0,1\}^*$. Because $x, y \in Y$, it follows that $\mathbf{diff}(x0y) = \mathbf{diff}(x) + \mathbf{diff}(0) + \mathbf{diff}(y) = 1 + -1 + 1 = 1$. So, it remains to show that, for all prefixes v of $x0y$, $\mathbf{diff}(v) \le 1$. Suppose v is a prefix of $x0y$. We must show that $\mathbf{diff}(v) \le 1$. There are two cases to consider.

- Suppose v is a prefix of x. Because $x \in Y$, it follows that $\mathbf{diff}(v) \le 1$.
- Suppose $v = x0u$, for some prefix u of y. Because $y \in Y$, it follows that $\mathbf{diff}(u) \le 1$. Hence, because $x \in Y$, we have that $\mathbf{diff}(v) = \mathbf{diff}(x0u) = \mathbf{diff}(x) + \mathbf{diff}(0) + \mathbf{diff}(u) = 1 + -1 + \mathbf{diff}(u) = \mathbf{diff}(u) \le 1$.

(3) Suppose $x, y \in X$ and assume the inductive hypothesis: $x, y \in Y$. We must show that $0xy \in Y$. Because $x, y \in Y \subseteq \{0,1\}^*$ and $0 \in \{0,1\}^*$, we have that $0xy \in \{0,1\}^*$. Because $x, y \in Y$, it follows that $\mathbf{diff}(0xy) = \mathbf{diff}(0) + \mathbf{diff}(x) + \mathbf{diff}(y) = -1 + 1 + 1 = 1$. So, it remains to show that, for all prefixes v of $0xy$, $\mathbf{diff}(v) \le 1$. Suppose v is a prefix of $0xy$. We must show that $\mathbf{diff}(v) \le 1$. There are three cases to consider.

- Suppose $v = \%$. Thus $\mathbf{diff}(v) = \mathbf{diff}(\%) = 0 \le 1$.
- Suppose $v = 0u$, for some prefix u of x. Because $x \in Y$, it follows that $\mathbf{diff}(u) \le 1$. Hence $\mathbf{diff}(v) = \mathbf{diff}(0u) = \mathbf{diff}(0) + \mathbf{diff}(u) = -1 + \mathbf{diff}(u) \le -1 + 1 = 0 \le 1$.
- Suppose $v = 0xu$, for some prefix u of y. Because $y \in Y$, it follows that $\mathbf{diff}(u) \le 1$. Hence, because $x \in Y$, we have that $\mathbf{diff}(v) = \mathbf{diff}(0xu) = \mathbf{diff}(0) + \mathbf{diff}(x) + \mathbf{diff}(u) = -1 + 1 + \mathbf{diff}(u) = \mathbf{diff}(u) \le 1$.

(b) First we prove a useful lemma.

Lemma ES1.4.1
For all $w \in \{0,1\}^*$, if $\mathbf{diff}(w) \ge 1$, then there are $x, y \in \{0,1\}^*$ such that $w = xy$, $x \in Y$ and $\mathbf{diff}(y) = \mathbf{diff}(w) - 1$.

Proof. Suppose $w \in \{0,1\}^*$ and suppose $\mathbf{diff}(w) \ge 1$. Let x be the shortest prefix of w such that $\mathbf{diff}(x) \ge 1$. (Such an x exists, because w is a prefix of itself with a positive diff.) Let $y \in \{0,1\}^*$ be such that $w = xy$. Because $\mathbf{diff}(x) \ge 1$, we have that $x \ne \%$. Thus $x = zb$, for some $z \in \{0,1\}^*$ and $b \in \{0,1\}$. Hence $w = xy = zby$. Because z is a shorter prefix of w than x, it follows that, (†) for all prefixes v of z, $\mathbf{diff}(v) \le 0$. In particular, $\mathbf{diff}(z) \le 0$.

Suppose, toward a contradiction, that $b = \mathsf{0}$. Since $\mathbf{diff}(z) + -1 = \mathbf{diff}(z) + \mathbf{diff}(\mathsf{0}) = \mathbf{diff}(z\mathsf{0}) = \mathbf{diff}(zb) = \mathbf{diff}(x) \geq 1$, we have that $\mathbf{diff}(z) \geq 2$—contradiction. Thus $b = \mathsf{1}$, so that $x = zb = z\mathsf{1}$.

Since $\mathbf{diff}(z) + 1 = \mathbf{diff}(z) + \mathbf{diff}(\mathsf{1}) = \mathbf{diff}(z\mathsf{1}) = \mathbf{diff}(x) \geq 1$, we have that $\mathbf{diff}(z) \geq 0$. But $\mathbf{diff}(z) \leq 0$, and thus $\mathbf{diff}(z) = 0$. Thus $\mathbf{diff}(x) = \mathbf{diff}(z\mathsf{1}) = \mathbf{diff}(z) + \mathbf{diff}(\mathsf{1}) = \mathbf{diff}(z) + 1 = 0 + 1 = 1$. To see that $x \in Y$, it remains to show that, for all prefixes v of x, $\mathbf{diff}(v) \leq 1$. Suppose v is a prefix of x. We must show that $\mathbf{diff}(v) \leq 1$. Since $x = z\mathsf{1}$, there are two cases to consider.

- Suppose v is a prefix of z. Thus $\mathbf{diff}(v) \leq 0 \leq 1$ by (†).
- Suppose $v = z\mathsf{1}$. Then $v = z\mathsf{1} = x$, so that $\mathbf{diff}(v) = \mathbf{diff}(x) = 1 \leq 1$.

Finally, we have that $\mathbf{diff}(w) = \mathbf{diff}(xy) = \mathbf{diff}(x) + \mathbf{diff}(y) = 1 + \mathbf{diff}(y)$, so that $\mathbf{diff}(y) = \mathbf{diff}(w) - 1$. □

Now, we use our lemma to prove that $Y \subseteq X$. Since $Y \subseteq \{\mathsf{0}, \mathsf{1}\}^*$, it will suffice to show that, for all $w \in \{\mathsf{0}, \mathsf{1}\}^*$,

$$\text{if } w \in Y, \text{ then } w \in X.$$

We proceed by strong string induction. Suppose $w \in \{\mathsf{0}, \mathsf{1}\}^*$, and assume the inductive hypothesis: for all $x \in \{\mathsf{0}, \mathsf{1}\}^*$, if $|x| < |w|$, then

$$\text{if } x \in Y, \text{ then } x \in X.$$

We must show that

$$\text{if } w \in Y, \text{ then } w \in X.$$

Suppose $w \in Y$. We must show that $w \in X$. There are three cases to consider.

- Suppose $w = \%$. Because $\% = w \in Y$, we have that $0 = \mathbf{diff}(\%) = 1$—contradiction. Thus $w \in X$.
- Suppose $w = \mathsf{0}t$, for some $t \in \{\mathsf{0}, \mathsf{1}\}^*$. Because $-1 + \mathbf{diff}(t) = \mathbf{diff}(\mathsf{0}) + \mathbf{diff}(t) = \mathbf{diff}(\mathsf{0}t) = \mathbf{diff}(w) = 1$, we have that $\mathbf{diff}(t) = 2$. Since $\mathbf{diff}(t) \geq 1$, Lemma ES1.4.1 tells us that $t = xy$ for some $x, y \in \{\mathsf{0}, \mathsf{1}\}^*$ such that $x \in Y$ and $\mathbf{diff}(y) = \mathbf{diff}(t) - 1$. Hence $\mathbf{diff}(y) = \mathbf{diff}(t) - 1 = 2 - 1 = 1$ and $w = \mathsf{0}t = \mathsf{0}xy$.

  To see that $y \in Y$, it remains to show that, for all prefixes v of y, $\mathbf{diff}(v) \leq 1$. Suppose v is a prefix of y. We must show that $\mathbf{diff}(v) \leq 1$. Since v is a prefix of y, we have that $\mathsf{0}xv$ is a prefix of $\mathsf{0}xy = w$. But $w \in Y$, and thus $\mathbf{diff}(v) = -1 + 1 + \mathbf{diff}(v) = \mathbf{diff}(\mathsf{0}) + \mathbf{diff}(x) + \mathbf{diff}(v) = \mathbf{diff}(\mathsf{0}xv) \leq 1$.

  Summarizing, we have that $w = \mathsf{0}xy$ and $x, y \in Y$. Because $|x| < |w|$ and $|y| < |w|$, the inductive hypothesis tells us that $x, y \in X$. Thus, by Part (3) of the definition of X, we have that $w = \mathsf{0}xy \in X$.
- Suppose $w = \mathsf{1}t$, for some $t \in \{\mathsf{0}, \mathsf{1}\}^*$. Because $1 + \mathbf{diff}(t) = \mathbf{diff}(\mathsf{1}) + \mathbf{diff}(t) = \mathbf{diff}(\mathsf{1}t) = \mathbf{diff}(w) = 1$, we have that $\mathbf{diff}(t) = 0$. There are three subcases to consider.
  - Suppose $t = \%$. Then $w = \mathsf{1}t = \mathsf{1}\% = \mathsf{1} \in X$, by Part (1) of the definition of X.

- Suppose $t = 0y$ for some $y \in \{0, 1\}^*$. Thus $w = 1t = 10y$. By Part (1) of the definition of X, we have that $1 \in X$. Furthermore, $\mathbf{diff}(y) = 1 + -1 + \mathbf{diff}(y) = \mathbf{diff}(1) + \mathbf{diff}(0) + \mathbf{diff}(y) = \mathbf{diff}(10y) = \mathbf{diff}(w) = 1$.

  To see that $y \in Y$, it remains to show that, for all prefixes v of y, $\mathbf{diff}(v) \leq 1$. Suppose v is a prefix of y. We must show that $\mathbf{diff}(v) \leq 1$. Since v is a prefix of y, we have that $10v$ is a prefix of $10y = w$. But $w \in Y$, and thus $\mathbf{diff}(v) = 1 + -1 + \mathbf{diff}(v) = \mathbf{diff}(1) + \mathbf{diff}(0) + \mathbf{diff}(v) = \mathbf{diff}(10v) \leq 1$.

  Summarizing, we have that $w = 10y$, $1 \in X$ and $y \in Y$. Because $|y| < |w|$, the inductive hypothesis tells us that $y \in X$. Hence $1, y \in X$, so that, by Part (2) of the definition of X, we have that $w = 10y \in X$.

- Suppose $t = 1y$ for some $y \in \{0, 1\}^*$. Then $w = 1t = 11y$, so that 11 is a prefix of w. But $w \in Y$, and thus $2 = 1 + 1 = \mathbf{diff}(1) + \mathbf{diff}(1) = \mathbf{diff}(11) \leq 1$—contradiction. Thus $w \in X$.