CIS 570 — Introduction to Formal Language Theory — Fall 2007

Exercise Set 3

Model Answers

Exercise 1

(a)

$$1 + (\% + 2 + 12)(012)^*(\% + 0 + 01)$$

(b) Let

$$Y = \{1\} \cup \{\%, 2, 12\}\{012\}^*\{\%, 0, 01\}.$$

Since $L(\alpha) = Y$, it will suffice to show that $Y = X$.

Let

$$A = \{\%\} \cup \{\, w \in X \mid 2 \text{ is a suffix of } w \,\},$$
$$B = \{\%\} \cup \{\, w \in X \mid 0 \text{ is a prefix of } w \text{ and } 2 \text{ is a suffix of } w \,\}.$$

Next, we prove a lemma that will help us show that $Y \subseteq X$.

Lemma ES3.1.1

(1) $\{012\}B \subseteq B$.

(2) $\{012\}^* \subseteq B$.

(3) $\{\%, 2, 12\}B \subseteq A$.

(4) $A\{\%, 0, 01\} \subseteq X$.

Proof.

(1) Suppose $w \in \{012\}B$. We must show that $w \in B$. Since $w \in \{012\}B$, we have that $w = 012x$, for some $x \in B$. Since $x \in B$, there are two cases to consider. If $x = \%$, then $w = 012$, so that $w \in B$. So, suppose $x \in X$, 0 is a prefix of x, and 2 is a suffix of x. Since $x \in X$ and 0 is a prefix of x, we have that $w = 012x \in X$. Clearly, 0 is a prefix of $012x = w$. And, since 2 is a suffix of x, we have that 2 is a suffix of $012x = w$. Hence $w \in B$.

(2) It will suffice to show that, for all $n \in \mathbb{N}$, $\{012\}^n \subseteq B$. We proceed by mathematical induction.

(Basis Step) We have that $\{012\}^0 = \{\%\} \subseteq B$.

(Inductive Step) Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: $\{012\}^n \subseteq B$. Then

$$\begin{aligned} \{012\}^{n+1} &= \{012\}\{012\}^n \\ &\subseteq \{012\}B && \text{(inductive hypothesis)} \\ &\subseteq B && \text{(Part (1))}. \end{aligned}$$

(3) Suppose $w \in \{\%, 2, 12\}B$. We must show that $w \in A$. Since $w \in \{\%, 2, 12\}B$, we have that $w = xy$, for some $x \in \{\%, 2, 12\}$ and $y \in B$. Because $x \in \{\%, 2, 12\}$, there are three cases to consider.

- Suppose $x = \%$. Then $w = xy = \%y = y \in B \subseteq A$.
- Suppose $x = 2$. Then $w = xy = 2y$. Since $y \in B$, there are two subcases to consider. If $y = \%$, then $w = 2y = 2 \in A$. Otherwise, $y \in X$, 0 is a prefix of y, and 2 is a suffix of y. Thus $w = 2y \in X$ and 2 is a suffix of $2y = w$. Hence $w \in A$.
- Suppose $x = 12$. Then $w = xy = 12y$. Since $y \in B$, there are two subcases to consider. If $y = \%$, then $w = 12y = 12 \in A$. Otherwise, $y \in X$, 0 is a prefix of y, and 2 is a suffix of y. Thus $w = 12y \in X$ and 2 is a suffix of $12y = w$. Hence $w \in A$.

(4) Suppose $w \in A\{\%, 0, 01\}$. We must show that $w \in X$. Since $w \in A\{\%, 0, 01\}$, we have that $w = xy$, for some $x \in A$ and $y \in \{\%, 0, 01\}$. Since $y \in \{\%, 0, 01\}$, there are three cases to consider.

- Suppose $y = \%$. Then $w = xy = x\% = x \in A \subseteq X$.
- Suppose $y = 0$. Then $w = xy = x0$. Since $x \in A$, there are two subcases to consider. If $x = \%$, then $w = x0 = 0 \in X$. Otherwise, $x \in X$ and 2 is a suffix of x, so that $w = x0 \in X$.
- Suppose $y = 01$. Then $w = xy = x01$. Since $x \in A$, there are two subcases to consider. If $x = \%$, then $w = x01 = 01 \in X$. Otherwise, $x \in X$ and 2 is a suffix of x, so that $w = x01 \in X$.

□

Now we can use the preceding lemma to show that $Y \subseteq X$. We have that

$$
\begin{aligned}
\{\%, 2, 12\}\{012\}^*\{\%, 0, 01\} &\subseteq \{\%, 2, 12\}B\{\%, 0, 01\} && \text{(Lemma ES3.1.1(2))} \\
&\subseteq A\{\%, 0, 01\} && \text{(Lemma ES3.1.1(3))} \\
&\subseteq X && \text{(Lemma ES3.1.1(4))}
\end{aligned}
$$

Furthermore, $1 \in X$, so that

$$
\begin{aligned}
Y &= \{1\} \cup \{\%, 2, 12\}\{012\}^*\{\%, 0, 01\} \\
&\subseteq X \cup X \\
&= X.
\end{aligned}
$$

Hence $Y \subseteq X$.

Finally, we show that $X \subseteq Y$. Since $X \subseteq \{0, 1, 2\}^*$, it will suffice to show that, for all $w \in \{0, 1, 2\}^*$,

$$\text{if } w \in X, \text{ then } w \in Y.$$

We proceed by strong string induction. Suppose $w \in \{0, 1, 2\}^*$, and assume the inductive hypothesis: for all $x \in \{0, 1, 2\}^*$, if $|x| < |w|$, then

$$\text{if } x \in X, \text{ then } x \in Y.$$

We must show that

$$\text{if } w \in X, \text{ then } w \in Y.$$

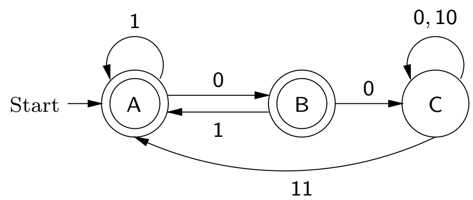
Suppose $w \in X$. We must show that $w \in Y$. There are four cases to consider.

- Suppose $w = \%$. Then $w = \%\%\% \in \{\%, 2, 12\}\{012\}^*\{\%, 0, 01\} \subseteq Y$.
- Suppose $w = 0x$, for some $x \in \{0, 1, 2\}^*$. If $x = \%$, then $w = 0 = \%\%0 \in \{\%, 2, 12\}\{012\}^*\{\%, 0, 01\} \subseteq Y$. So, suppose $x \neq \%$. Since $0x = w \in X$, it follows that $x = 1y$, for some $y \in \{0, 1, 2\}^*$. Thus $w = 01y$. If $y = \%$, then $w = 01 = \%\%(01) \in \{\%, 2, 12\}\{012\}^*\{\%, 0, 01\} \subseteq Y$. So, suppose $y \neq \%$. Since $01y = w \in X$, it follows that $y = 2z$, for some $z \in \{0, 1, 2\}^*$. Thus $w = 012z$. Because $012z = w \in X$, we have that $12z \in X$. Furthermore, since $|12z| < |w|$, the inductive hypothesis tells us that $12z \in Y$. Because $12z \neq 1$, we have that $12z = tuv$, for some $t \in \{\%, 2, 12\}$, $u \in \{012\}^*$ and $v \in \{\%, 0, 01\}$. Since $t \in \{\%, 2, 12\}$, there are three subcases to consider.
  - Suppose $t = \%$. Thus $12z = tuv = \%uv = uv$. Because $u \in \{012\}^*$, it follows that $u = \%$. Hence $12z = uv = \%v = v$. But $v \in \{\%, 0, 01\}$, and none of the elements of $\{\%, 0, 01\}$ begin with 1—contradiction. Thus $w \in Y$.
  - Suppose $t = 2$. Since $12z = tuv$, it follows that $1 = 2$—contradiction. Thus $w \in Y$.
  - Suppose $t = 12$. Thus $w = 012z = 0(12z) = 0tuv = (012)uv = \%(012)uv \in \{\%, 2, 12\}\{012\}\{012\}^*\{\%, 0, 01\} \subseteq \{\%, 2, 12\}\{012\}^*\{\%, 0, 01\} \subseteq Y$.
- Suppose $w = 1x$, for some $x \in \{0, 1, 2\}^*$. If $x = \%$, then $w = 1x = 1 \in Y$. So, suppose $x \neq \%$. Since $1x = w \in X$, it follows that $x = 2y$, for some $y \in \{0, 1, 2\}^*$. Thus $w = 12y$. Because $12y = w \in X$, we have that $2y \in X$. Furthermore, since $|2y| < |w|$, the inductive hypothesis tells us that $2y \in Y$. Because $2y \neq 1$, we have that $2y = tuv$, for some $t \in \{\%, 2, 12\}$, $u \in \{012\}^*$ and $v \in \{\%, 0, 01\}$. Since $t \in \{\%, 2, 12\}$, there are three subcases to consider.
  - Suppose $t = \%$. Thus $2y = tuv = \%uv = uv$. Because $u \in \{012\}^*$, it follows that $u = \%$. Hence $2y = uv = \%v = v$. But $v \in \{\%, 0, 01\}$, and none of the elements of $\{\%, 0, 01\}$ begin with 2—contradiction. Thus $w \in Y$.
  - Suppose $t = 2$. Thus $w = 12y = 1(2y) = 1tuv = (12)uv \in \{\%, 2, 12\}\{012\}^*\{\%, 0, 01\} \subseteq Y$.
  - Suppose $t = 12$. Since $2y = tuv$, it follows that $2 = 1$—contradiction. Thus $w \in Y$.
- Suppose $w = 2x$, for some $x \in \{0, 1, 2\}^*$. If $x = \%$, then $w = 2x = 2 = 2\%\% \in \{\%, 2, 12\}\{012\}^*\{\%, 0, 01\} \subseteq Y$. So, suppose $x \neq \%$. Since $2x = w \in X$, it follows that $x = 0y$, for some $y \in \{0, 1, 2\}^*$. Thus $w = 20y$. Because $20y = w \in X$, we have that $0y \in X$. Furthermore, since $|0y| < |w|$, the inductive hypothesis tells us that $0y \in Y$. Because $0y \neq 1$, we have that $0y = tuv$, for some $t \in \{\%, 2, 12\}$, $u \in \{012\}^*$ and $v \in \{\%, 0, 01\}$. Since $0y = tuv$ and $t \in \{\%, 2, 12\}$, it follows that $t = \%$. Thus $w = 20y = 2(0y) = 2tuv = 2uv \in \{\%, 2, 12\}\{012\}^*\{\%, 0, 01\} \subseteq Y$.

Because $Y \subseteq X \subseteq Y$, we have that $Y = X$.

Exercise 2

(a)



(b) First, we put the following text, which describes M in Forlan's syntax, in the file `es3-ex2-fa`:

```
{states}
A, B, C
{start state}
A
{accepting states}
A, B
{transitions}
A, 0 -> B; A, 1 -> A;
B, 0 -> C; B, 1 -> A;
C, 0 -> C; C, 10 -> C; C, 11 -> A
```

Next, we load M into Forlan, and call it `fa`:

```
- val fa = FA.input "es3-ex2-fa";
val fa = - : fa
```

Next, we find minimum-length labeled paths explaining why some strings in X are accepted by `fa` (M):

```
- val findAcceptingLP = FA.findAcceptingLP fa;
val findAcceptingLP = fn : str -> lp
- fun test s =
=       (print(s ^ ":\n");
=        LP.output("", findAcceptingLP(Str.fromString s)));
val test = fn : string -> unit
- app test
=     ["%", "1", "0", "11", "011", "110", "0101", "0011", "00110",
=      "01000100110", "0011010", "00011010", "00010100011"];
%:
A
1:
A, 1 => A
0:
A, 0 => B
11:
A, 1 => A, 1 => A
```

```
011:
A, 0 => B, 1 => A, 1 => A
110:
A, 1 => A, 1 => A, 0 => B
0101:
A, 0 => B, 1 => A, 0 => B, 1 => A
0011:
A, 0 => B, 0 => C, 11 => A
00110:
A, 0 => B, 0 => C, 11 => A, 0 => B
01000100110:
A, 0 => B, 1 => A, 0 => B, 0 => C, 0 => C, 10 => C, 0 => C, 11 => A, 0 => B
0011010:
A, 0 => B, 0 => C, 11 => A, 0 => B, 1 => A, 0 => B
00011010:
A, 0 => B, 0 => C, 0 => C, 11 => A, 0 => B, 1 => A, 0 => B
00010100011:
A, 0 => B, 0 => C, 0 => C, 10 => C, 10 => C, 0 => C, 0 => C, 11 => A
val it = () : unit
```

Finally, we check that some strings that are not in X are not accepted by `fa` (M):

```
- val accepted = FA.accepted fa;
val accepted = fn : str -> bool
- fun test s = (s, accepted(Str.fromString s));
val test = fn : string -> string * bool
- map test
=     ["00", "001", "0010", "000101", "000", "0100101", "1100",
=      "1010001100", "1000011010010"];
val it =
  [("00",false),("001",false),("0010",false),("000101",false),("000",false),
   ("0100101",false),("1100",false),("1010001100",false),
   ("1000011010010",false)] : (string * bool) list
```

(c) First we prove some basic properties about X.

Lemma ES3.2.1

(1) For all $w \in \{0,1\}^*$, if 00 is not a substring of w, then $w \in X$.

(2) $\{0,1\}^*\{11\} \subseteq X$.

(3) $X\{1\} \subseteq X$.

Proof.

(1) Suppose $w \in \{0,1\}^*$ and 00 is not a substring of w. To see that $w \in X$, suppose $x, y \in \{0,1\}^*$ and $w = x00y$. We must show that 11 is a substring of y. Then 00 is a substring of w—contradiction. Thus 11 is a substring of y.

(2) Suppose $w \in \{0,1\}^*\{11\}$. Hence $w = x11$ for some $x \in \{0,1\}^*$. To see that $w \in X$, suppose $u, v \in \{0,1\}^*$ and $w = u00v$. We must show that 11 is a substring of v. We have that $u00v = w = x11$. Thus 11 is a suffix, and thus a substring, of v.

(3) Suppose $w \in X\{1\}$. Thus $w = x1$ for some $x \in X$. To see that $w \in X$, suppose $u, v \in \{0,1\}^*$ and $w = u00v$. We must show that 11 is a substring of v. We have that $u00v = w = x1$. Thus $v = t1$ for some $t \in \{0,1\}^*$. Because $x1 = u00v = u00t1$, we have that $x = u00t$. Because $x \in X$, it follows that 11 is a substring of t. But $v = t1$, and thus 11 is a substring of v.

□

By Lemma ES3.2.1(1), we have that $\%, 0 \in X$. Define languages A, B and C by:

$$A = \{\, w \in X \mid 0 \text{ is not a suffix of } w \,\},$$
$$B = \{\, w \in X \mid 0 \text{ is a suffix of } w \,\},$$
$$C = \{\, w \in \{0,1\}^* \mid w \notin X \text{ and } 0 \text{ is a suffix of } w \,\}.$$

It is easy to see that $X = A \cup B$. Next, we prove a useful lemma that is derived from the definition of M.

Lemma ES3.2.2
For all $w \in \{0,1\}^*$:

(A) $w \in A$ iff $w = \%$, or $w = x1$ for some $x \in A$, or $w = x1$ for some $x \in B$, or $w = x11$ for some $x \in C$;

(B) $w \in B$ iff $w = x0$ for some $x \in A$;

(C) $w \in C$ iff $w = x0$ for some $x \in B$, or $w = x0$ for some $x \in C$, or $w = x10$ for some $x \in C$.

Proof.

(A) (“only if” direction) Suppose $w \in A$. Thus $w \in X$ and 0 is not a suffix of w. If $w = \%$, then we are done, so suppose $w \neq \%$. Because 0 is not a suffix of w, we have that $w = x1$ for some $x \in \{0,1\}^*$. There are two cases to consider.

- Suppose $x \in X$. Because $X = A \cup B$, we have that $x \in A$ or $x \in B$, and thus either $w = x1$ and $x \in A$, or $w = x1$ and $x \in B$.
- Suppose $x \notin X$. Thus $x = u00v$ for some $u, v \in \{0,1\}^*$ such that 11 is not a substring of v. Hence $w = x1 = u00v1$. Because $u00v1 = w \in X$, it follows that 11 is a substring of $v1$. But 11 is not a substring of v, and thus 1 is a suffix of v. Thus $v = t1$ for some $t \in \{0,1\}^*$, so that $x = u00v = u00t1$. Hence $w = x1 = u00t11$. Because 11 is not a substring of $v = t1$, it follows that 11 is not a substring of t and 1 is not a suffix of t. Thus $u00t \notin X$. And, either $t = \%$ or 0 is a suffix of t. Hence we have that 0 is a suffix of $u00t$. Thus $w = (u00t)11$ and $u00t \in C$.

(“if” direction) Suppose $w = \%$, or $w = x1$ for some $x \in A$, or $w = x1$ for some $x \in B$, or $w = x11$ for some $x \in C$. There are four cases to consider.

- Suppose $w = \%$. Then $w = \% \in X$. And $\mathsf{0}$ is not a substring of $\% = w$, completing the proof that $w \in A$.
- Suppose $w = x\mathsf{1}$ for some $x \in A$. Thus $x \in X$. By Lemma ES3.2.1(3), we have that $w = x\mathsf{1} \in X\{\mathsf{1}\} \subseteq X$. And $\mathsf{0}$ is not a suffix of $x\mathsf{1} = w$, completing the proof that $w \in A$.
- Suppose $w = x\mathsf{1}$ for some $x \in B$. Thus $x \in X$. By Lemma ES3.2.1(3), we have that $w = x\mathsf{1} \in X\{\mathsf{1}\} \subseteq X$. And $\mathsf{0}$ is not a suffix of $x\mathsf{1} = w$, completing the proof that $w \in A$.
- Suppose $w = x\mathsf{11}$ for some $x \in C$. By Lemma ES3.2.1(2), we have that $w = x\mathsf{11} \in \{\mathsf{0},\mathsf{1}\}^*\{\mathsf{11}\} \subseteq X$. And $\mathsf{0}$ is not a suffix of $x\mathsf{11} = w$, completing the proof that $w \in A$.

(B) ("only if" direction) Suppose $w \in B$. Thus $w \in X$ and $\mathsf{0}$ is a suffix of w. Hence $w = x\mathsf{0}$ for some $x \in \{\mathsf{0},\mathsf{1}\}^*$.

Suppose, toward a contradiction, that $\mathsf{0}$ is a suffix of x. Then $x = y\mathsf{0}$ for some $y \in \{\mathsf{0},\mathsf{1}\}^*$, so that $w = x\mathsf{0} = y\mathsf{00}\%$. But then $w \notin X$—contradiction. Thus $\mathsf{0}$ is not a suffix of x.

Suppose, toward a contradiction, that $x \notin X$. Thus $x = u\mathsf{00}v$ for some $u, v \in \{\mathsf{0},\mathsf{1}\}^*$ such that $\mathsf{11}$ is not a substring of v. Because $u\mathsf{00}v\mathsf{0} = x\mathsf{0} = w \in X$, we have that $\mathsf{11}$ is a substring of $v\mathsf{0}$, so that $\mathsf{11}$ is a substring of v—contradiction. Thus $x \in X$, completing the proof that $x \in A$.

Hence $w = x\mathsf{0}$ and $x \in A$.

("if" direction) Suppose $w = x\mathsf{0}$ for some $x \in A$. Thus $x \in X$ and $\mathsf{0}$ is not a suffix of x.

To see that $w \in X$, suppose that $u, v \in \{\mathsf{0},\mathsf{1}\}^*$ and $w = u\mathsf{00}v$. We must show that $\mathsf{11}$ is a substring of v. We have that $x\mathsf{0} = w = u\mathsf{00}v$. Because $\mathsf{0}$ is not a suffix of x, we have that $v \neq \%$. Thus $v = t\mathsf{0}$ for some $t \in \{\mathsf{0},\mathsf{1}\}^*$. Hence $x\mathsf{0} = u\mathsf{00}v = u\mathsf{00}t\mathsf{0}$, so that $x = u\mathsf{00}t$. Because $x \in X$, it follows that $\mathsf{11}$ is a substring of t. But $v = t\mathsf{0}$, and thus $\mathsf{11}$ is a substring of v.

Finally, $\mathsf{0}$ is a suffix of $x\mathsf{0} = w$, completing the proof that $w \in B$.

(C) ("only if" direction) Suppose $w \in C$. Thus $w \in \{\mathsf{0},\mathsf{1}\}^*$, $w \notin X$ and $\mathsf{0}$ is a suffix of w, so that $w = x\mathsf{0}$ for some $x \in \{\mathsf{0},\mathsf{1}\}^*$. If $x = \%$, then $w = \mathsf{0} \in X$—contradiction. Thus $x \neq \%$. There are two cases to consider.

- Suppose $x = y\mathsf{0}$ for some $y \in \{\mathsf{0},\mathsf{1}\}^*$. Thus $\mathsf{0}$ is a suffix of x. There are two subcases to consider.
  - Suppose $x \in X$. Thus $w = x\mathsf{0}$ and $x \in B$.
  - Suppose $x \notin X$. Thus $w = x\mathsf{0}$ and $x \in C$.
- Suppose $x = y\mathsf{1}$ for some $y \in \{\mathsf{0},\mathsf{1}\}^*$. Hence $w = x\mathsf{0} = y\mathsf{10}$. Because $w \notin X$, there are $u, v \in \{\mathsf{0},\mathsf{1}\}^*$ such that $w = u\mathsf{00}v$ and $\mathsf{11}$ is not a substring of v. Thus $y\mathsf{10} = w = u\mathsf{00}v$. Clearly $v \neq \%$, and so $\mathsf{0}$ is a suffix of v. And v cannot be $\mathsf{0}$, and thus $v = t\mathsf{10}$ for some $t \in \{\mathsf{0},\mathsf{1}\}^*$. Because $y\mathsf{10} = u\mathsf{00}v = u\mathsf{00}t\mathsf{10}$, we have that $y = u\mathsf{00}t$. Since $\mathsf{11}$ is not a substring of $v = t\mathsf{10}$, it follows that $\mathsf{11}$ is not a substring of t, and $\mathsf{1}$ is not a suffix of t. Hence $y = u\mathsf{00}t \notin X$. Because $\mathsf{1}$ is not a suffix of t, either $t = \%$ or $\mathsf{0}$ is a suffix of t. In either case, we have that $\mathsf{0}$ is suffix of $u\mathsf{00}t = y$, completing the proof that $y \in C$. Thus $w = y\mathsf{10}$ and $y \in C$.

("if" direction) Suppose $w = x\mathsf{0}$ for some $x \in B$, or $w = x\mathsf{0}$ for some $x \in C$, or $w = x\mathsf{10}$ for some $x \in C$. There are three cases to consider.

- Suppose $w = x\mathsf{0}$ for some $x \in B$. Thus $\mathsf{0}$ is a suffix of x, so that $\mathsf{00}$ is a suffix of $x\mathsf{0} = w$. Hence $w = u\mathsf{00}\%$ for some $u \in \{\mathsf{0},\mathsf{1}\}^*$, so that $w \notin X$. And $\mathsf{0}$ is a suffix of $x\mathsf{0} = w$, completing the proof that $w \in C$.
- Suppose $w = x\mathsf{0}$ for some $x \in C$. Thus $\mathsf{0}$ is a suffix of x, so that $\mathsf{00}$ is a suffix of $x\mathsf{0} = w$. Hence $w = u\mathsf{00}\%$ for some $u \in \{\mathsf{0},\mathsf{1}\}^*$, so that $w \notin X$. And $\mathsf{0}$ is a suffix of $x\mathsf{0} = w$, completing the proof that $w \in C$.
- Suppose $w = x\mathsf{10}$ for some $x \in C$. Thus $\mathsf{0}$ is a suffix of x and $x \notin X$, i.e., $x = u\mathsf{00}v$ for some $u, v \in \{\mathsf{0},\mathsf{1}\}^*$ such that $\mathsf{11}$ is not a substring of v. Because $\mathsf{0}$ is a suffix of x, and $x = u\mathsf{00}v$, it follows that $\mathsf{1}$ is not a suffix of v. Hence $\mathsf{11}$ is not a substring of $v\mathsf{10}$. But $w = x\mathsf{10} = u\mathsf{00}v\mathsf{10}$, and thus $w \notin X$. And $\mathsf{0}$ is a suffix of $x\mathsf{10} = w$, completing the proof that $w \in C$.

□

Now we prove a lemma that will allow us to establish that $L(M) \subseteq X$.

Lemma ES3.2.3
For all $w \in \{\mathsf{0},\mathsf{1}\}^*$:

(A) if $\mathsf{A} \in \Delta(\{\mathsf{A}\}, w)$, then $w \in A$;

(B) if $\mathsf{B} \in \Delta(\{\mathsf{A}\}, w)$, then $w \in B$;

(C) if $\mathsf{C} \in \Delta(\{\mathsf{A}\}, w)$, then $w \in C$.

Proof. We proceed by strong string induction. Suppose $w \in \{\mathsf{0},\mathsf{1}\}^*$, and assume the inductive hypothesis: for all $x \in \{\mathsf{0},\mathsf{1}\}^*$, if $|x| < |w|$, then:

(A) if $\mathsf{A} \in \Delta(\{\mathsf{A}\}, x)$, then $x \in A$;

(B) if $\mathsf{B} \in \Delta(\{\mathsf{A}\}, x)$, then $x \in B$;

(C) if $\mathsf{C} \in \Delta(\{\mathsf{A}\}, x)$, then $x \in C$.

We must show that:

(A) if $\mathsf{A} \in \Delta(\{\mathsf{A}\}, w)$, then $w \in A$;

(B) if $\mathsf{B} \in \Delta(\{\mathsf{A}\}, w)$, then $w \in B$;

(C) if $\mathsf{C} \in \Delta(\{\mathsf{A}\}, w)$, then $w \in C$.

We proceed as follows.

(A) Suppose $\mathsf{A} \in \Delta(\{\mathsf{A}\}, w)$. We must show that $w \in A$. Since $\mathsf{A} \in \Delta(\{\mathsf{A}\}, w)$, there are two cases to consider.

- Suppose $\mathsf{A} = \mathsf{A}$ and $w = \%$. By Lemma ES3.2.2(A), $w \in A$.

- Suppose there are $q \in Q$ and $x, y \in \mathbf{Str}$ such that $w = xy$, $q \in \Delta(\{\mathsf{A}\}, x)$ and $(q, y, \mathsf{A}) \in T$. Since $(q, y, \mathsf{A}) \in T$, there are three cases to consider.
  - Suppose $q = \mathsf{A}$ and $y = \mathsf{1}$. Thus $\mathsf{A} \in \Delta(\{\mathsf{A}\}, x)$ and $w = xy = x\mathsf{1}$. Since $|x| < |w|$, Part (A) of the inductive hypothesis tells us that $x \in A$. Hence, by Lemma ES3.2.2(A), we have that $w \in A$.
  - Suppose $q = \mathsf{B}$ and $y = \mathsf{1}$. Thus $\mathsf{B} \in \Delta(\{\mathsf{A}\}, x)$ and $w = xy = x\mathsf{1}$. Since $|x| < |w|$, Part (B) of the inductive hypothesis tells us that $x \in B$. Hence, by Lemma ES3.2.2(A), we have that $w \in A$.
  - Suppose $q = \mathsf{C}$ and $y = \mathsf{11}$. Thus $\mathsf{C} \in \Delta(\{\mathsf{A}\}, x)$ and $w = xy = x\mathsf{11}$. Since $|x| < |w|$, Part (C) of the inductive hypothesis tells us that $x \in C$. Hence, by Lemma ES3.2.2(A), we have that $w \in A$.

(B) Suppose $\mathsf{B} \in \Delta(\{\mathsf{A}\}, w)$. We must show that $w \in B$. Since $\mathsf{B} \in \Delta(\{\mathsf{A}\}, w)$, there are $q \in Q$ and $x, y \in \mathbf{Str}$ such that $w = xy$, $q \in \Delta(\{\mathsf{A}\}, x)$ and $(q, y, \mathsf{B}) \in T$. Since $(q, y, \mathsf{B}) \in T$, we have that $q = \mathsf{A}$ and $y = \mathsf{0}$. Thus $\mathsf{A} \in \Delta(\{\mathsf{A}\}, x)$ and $w = xy = x\mathsf{0}$. Since $|x| < |w|$, Part (A) of the inductive hypothesis tells us that $x \in A$. Hence, by Lemma ES3.2.2(B), we have that $w \in B$.

(C) Suppose $\mathsf{C} \in \Delta(\{\mathsf{A}\}, w)$. We must show that $w \in C$. Since $\mathsf{C} \in \Delta(\{\mathsf{A}\}, w)$, there are $q \in Q$ and $x, y \in \mathbf{Str}$ such that $w = xy$, $q \in \Delta(\{\mathsf{A}\}, x)$ and $(q, y, \mathsf{C}) \in T$. Since $(q, y, \mathsf{C}) \in T$, there are three cases to consider.

- Suppose $q = \mathsf{B}$ and $y = \mathsf{0}$. Thus $\mathsf{B} \in \Delta(\{\mathsf{A}\}, x)$ and $w = xy = x\mathsf{0}$. Since $|x| < |w|$, Part (B) of the inductive hypothesis tells us that $x \in B$. Hence, by Lemma ES3.2.2(C), we have that $w \in C$.
- Suppose $q = \mathsf{C}$ and $y = \mathsf{0}$. Thus $\mathsf{C} \in \Delta(\{\mathsf{A}\}, x)$ and $w = xy = x\mathsf{0}$. Since $|x| < |w|$, Part (C) of the inductive hypothesis tells us that $x \in C$. Hence, by Lemma ES3.2.2(C), we have that $w \in C$.
- Suppose $q = \mathsf{C}$ and $y = \mathsf{10}$. Thus $\mathsf{C} \in \Delta(\{\mathsf{A}\}, x)$ and $w = xy = x\mathsf{10}$. Since $|x| < |w|$, Part (C) of the inductive hypothesis tells us that $x \in C$. Hence, by Lemma ES3.2.2(C), we have that $w \in C$.

□

Now, we use the preceding lemma to show that $L(M) \subseteq X$. Suppose $w \in L(M)$. We must show that $w \in X$. Because $w \in L(M)$, we have that $\Delta(\{\mathsf{A}\}, w) \cap \{\mathsf{A}, \mathsf{B}\} \neq \emptyset$, so that $\mathsf{A} \in \Delta(\{\mathsf{A}\}, w)$ or $\mathsf{B} \in \Delta(\{\mathsf{A}\}, w)$. Since $\mathbf{alphabet}(M) = \{\mathsf{0}, \mathsf{1}\}$, we have that $w \in \{\mathsf{0}, \mathsf{1}\}^*$. Thus, we have that Parts (A)–(C) of Lemma ES3.2.3 hold. By Parts (A) and (B), it follows that $w \in A$ or $w \in B$. But $A \subseteq X$ and $B \subseteq X$, and thus $w \in X$.

Next, we prove a lemma that will allow us to establish that $X \subseteq L(M)$.

Lemma ES3.2.4
For all $w \in \{\mathsf{0}, \mathsf{1}\}^*$:

(A) if $w \in A$, then $\mathsf{A} \in \Delta(\{\mathsf{A}\}, w)$;

(B) if $w \in B$, then $\mathsf{B} \in \Delta(\{\mathsf{A}\}, w)$;

(C) if $w \in C$, then $\mathsf{C} \in \Delta(\{\mathsf{A}\}, w)$.

Proof. We proceed by strong string induction. Suppose $w \in \{0,1\}^*$, and assume the inductive hypothesis: for all $x \in \{0,1\}^*$, if $|x| < |w|$, then:

(A) if $x \in A$, then $\mathsf{A} \in \Delta(\{\mathsf{A}\}, x)$;

(B) if $x \in B$, then $\mathsf{B} \in \Delta(\{\mathsf{A}\}, x)$;

(C) if $x \in C$, then $\mathsf{C} \in \Delta(\{\mathsf{A}\}, x)$.

We must show that:

(A) if $w \in A$, then $\mathsf{A} \in \Delta(\{\mathsf{A}\}, w)$;

(B) if $w \in B$, then $\mathsf{B} \in \Delta(\{\mathsf{A}\}, w)$;

(C) if $w \in C$, then $\mathsf{C} \in \Delta(\{\mathsf{A}\}, w)$.

We proceed as follows.

(A) Suppose $w \in A$. We must show that $\mathsf{A} \in \Delta(\{\mathsf{A}\}, w)$. By Lemma ES3.2.2(A), there are four cases to consider.

- Suppose $w = \%$. We have that $\mathsf{A} \in \Delta(\{\mathsf{A}\}, \%)$, and thus that $\mathsf{A} \in \Delta(\{\mathsf{A}\}, w)$.
- Suppose $w = x1$ for some $x \in A$. Since $|x| < |w|$, Part (A) of the inductive hypothesis tells us that $\mathsf{A} \in \Delta(\{\mathsf{A}\}, x)$. Since $(\mathsf{A}, 1, \mathsf{A}) \in T$, we have that $\mathsf{A} \in \Delta(\{\mathsf{A}\}, 1)$. Hence $\mathsf{A} \in \Delta(\{\mathsf{A}\}, x1)$, i.e., $\mathsf{A} \in \Delta(\{\mathsf{A}\}, w)$.
- Suppose $w = x1$ for some $x \in B$. Since $|x| < |w|$, Part (B) of the inductive hypothesis tells us that $\mathsf{B} \in \Delta(\{\mathsf{A}\}, x)$. Since $(\mathsf{B}, 1, \mathsf{A}) \in T$, we have that $\mathsf{A} \in \Delta(\{\mathsf{B}\}, 1)$. Hence $\mathsf{A} \in \Delta(\{\mathsf{A}\}, x1)$, i.e., $\mathsf{A} \in \Delta(\{\mathsf{A}\}, w)$.
- Suppose $w = x11$ for some $x \in C$. Since $|x| < |w|$, Part (C) of the inductive hypothesis tells us that $\mathsf{C} \in \Delta(\{\mathsf{A}\}, x)$. Since $(\mathsf{C}, 11, \mathsf{A}) \in T$, we have that $\mathsf{A} \in \Delta(\{\mathsf{C}\}, 11)$. Hence $\mathsf{A} \in \Delta(\{\mathsf{A}\}, x11)$, i.e., $\mathsf{A} \in \Delta(\{\mathsf{A}\}, w)$.

(B) Suppose $w \in B$. We must show that $\mathsf{B} \in \Delta(\{\mathsf{A}\}, w)$. By Lemma ES3.2.2(B), we have that $w = x0$ for some $x \in A$. Since $|x| < |w|$, Part (A) of the inductive hypothesis tells us that $\mathsf{A} \in \Delta(\{\mathsf{A}\}, x)$. Since $(\mathsf{A}, 0, \mathsf{B}) \in T$, we have that $\mathsf{B} \in \Delta(\{\mathsf{A}\}, 0)$. Hence $\mathsf{B} \in \Delta(\{\mathsf{A}\}, x0)$, i.e., $\mathsf{B} \in \Delta(\{\mathsf{A}\}, w)$.

(C) Suppose $w \in C$. We must show that $\mathsf{C} \in \Delta(\{\mathsf{A}\}, w)$. By Lemma ES3.2.2(C), there are three cases to consider.

- Suppose $w = x0$ for some $x \in B$. Since $|x| < |w|$, Part (B) of the inductive hypothesis tells us that $\mathsf{B} \in \Delta(\{\mathsf{A}\}, x)$. Since $(\mathsf{B}, 0, \mathsf{C}) \in T$, we have that $\mathsf{C} \in \Delta(\{\mathsf{B}\}, 0)$. Hence $\mathsf{C} \in \Delta(\{\mathsf{A}\}, x0)$, i.e., $\mathsf{C} \in \Delta(\{\mathsf{A}\}, w)$.

- Suppose $w = x\mathsf{0}$ for some $x \in C$. Since $|x| < |w|$, Part (C) of the inductive hypothesis tells us that $\mathsf{C} \in \Delta(\{\mathsf{A}\}, x)$. Since $(\mathsf{C}, \mathsf{0}, \mathsf{C}) \in T$, we have that $\mathsf{C} \in \Delta(\{\mathsf{C}\}, \mathsf{0})$. Hence $\mathsf{C} \in \Delta(\{\mathsf{A}\}, x\mathsf{0})$, i.e., $\mathsf{C} \in \Delta(\{\mathsf{A}\}, w)$.
- Suppose $w = x\mathsf{10}$ for some $x \in C$. Since $|x| < |w|$, Part (C) of the inductive hypothesis tells us that $\mathsf{C} \in \Delta(\{\mathsf{A}\}, x)$. Since $(\mathsf{C}, \mathsf{10}, \mathsf{C}) \in T$, we have that $\mathsf{C} \in \Delta(\{\mathsf{C}\}, \mathsf{10})$. Hence $\mathsf{C} \in \Delta(\{\mathsf{A}\}, x\mathsf{10})$, i.e., $\mathsf{C} \in \Delta(\{\mathsf{A}\}, w)$.

□

Now, we use the preceding lemma to show that $X \subseteq L(M)$. Suppose $w \in X$. We must show that $w \in L(M)$. Since $X \subseteq \{\mathsf{0}, \mathsf{1}\}^*$, we have that Parts (A)–(C) of Lemma ES3.2.4 hold. There are two cases to consider.

- Suppose $\mathsf{0}$ is a suffix of w. Then $w \in B$, so that $\mathsf{B} \in \Delta(\{\mathsf{A}\}, w)$, by Part (B). Thus $\Delta(\{\mathsf{A}\}, w) \cap \{\mathsf{A}, \mathsf{B}\} \neq \emptyset$, showing that $w \in L(M)$.
- Suppose $\mathsf{0}$ is not a suffix of w. Then $w \in A$, so that $\mathsf{A} \in \Delta(\{\mathsf{A}\}, w)$, by Part (A). Thus $\Delta(\{\mathsf{A}\}, w) \cap \{\mathsf{A}, \mathsf{B}\} \neq \emptyset$, showing that $w \in L(M)$.

Because $L(M) \subseteq X \subseteq L(M)$, we have that $L(M) = X$. Finally, we must show that M has as few states as possible.

Lemma ES3.2.5
For all FAs N, if $L(N) = X$ and $|Q_N| \leq 2$, then, for all $q \in Q_N$, $\Delta(\{q\}, \%) \cap A_N \neq \emptyset$.

Proof. There are two cases to consider.

- Suppose $|Q_N| = 1$. Because $\% \in X = L(N)$, we have that $s_N \in A_N$. Because $s_N \in \Delta(\{s_N\}, \%)$, we have that $\Delta(\{s_N\}, \%) \cap A_N \neq \emptyset$.
- Suppose $|Q_N| = 2$. Let q be the non-start state of N.

  First, we show that $\Delta(\{s_N\}, \%) \cap A_N \neq \emptyset$. If $s_N \in A_N$, then $s_N \in \Delta(\{s_N\}, \%)$ and $s_N \in A_N$, and so we are done. So, suppose $s_N \notin A_N$. Because $\% \in L(N)$, we must have that $(s_N, \%, q) \in T$ and $q \in A_N$. Thus $q \in \Delta(\{s_N\}, \%)$ and $q \in A_N$, so that $\Delta(\{s_N\}, \%) \cap A_N \neq \emptyset$.

  It remains to show that $\Delta(\{q\}, \%) \cap A_N \neq \emptyset$. If $q \in A_N$, then this holds, so suppose $q \notin A_N$. Then $s_N \in A_N$, as otherwise N would accept nothing. If $(s_N, \mathsf{0}, s_N) \in T$, then $\%(\mathsf{00})\% = \mathsf{00} \in L(N) = X$—contradiction. Thus $(s_N, \mathsf{0}, s_N) \notin T$. Because $\mathsf{0} \in X = L(N)$, there must be a labeled path *lp* that is valid for N, starts and ends at s_N, and is labeled by $\mathsf{0}$. Since $(s_N, \mathsf{0}, s_N) \notin T$, *lp* must visit q on its way from s_N to s_N. If its first return to s_N uses label $\%$, then $(q, \%, s_N) \in T$, and thus $\Delta(\{q\}, \%) \cap A_N \neq \emptyset$. Otherwise, its first return to s_N uses label $\mathsf{0}$, so that $(q, \mathsf{0}, s_N) \in T$. But then $(s_N, \%, q) \in T$, as otherwise *lp* wouldn't have label $\mathsf{0}$. Thus $s_N \in \Delta(\{s_N\}, \%\mathsf{0})$, i.e., $s_N \in \Delta(\{s_N\}, \mathsf{0})$. But then $s_N \in \Delta(\{s_N\}, \mathsf{00})$, showing that $\mathsf{00} \in L(N) = X$—contradiction. Thus $\Delta(\{q\}, \%) \cap A_N \neq \emptyset$.

□

Lemma ES3.2.6
For all $n \in \mathbb{N}$, $\mathsf{00}(\mathsf{0}^n)\mathsf{11}$ is a length $n + 4$ element of X, and, for all prefixes v of $\mathsf{00}(\mathsf{0}^n)\mathsf{11}$, if $2 \leq |v| \leq n + 3$, then $v \notin X$.

Proof. Suppose $n \in \mathbb{N}$ and let $w = \mathsf{00}(\mathsf{0}^n)\mathsf{11}$. Clearly $|w| = 2+n+2 = n+4$. By Lemma ES3.2.1(2), we have that $w = \mathsf{00}(\mathsf{0}^n)\mathsf{11} \in \{\mathsf{0}, \mathsf{1}\}^*\{\mathsf{11}\} \subseteq X$.

For the last part, suppose v is a prefix of w such that $2 \leq |v| \leq n + 3$. There are two cases to consider.

- Suppose $v = \mathsf{000}^i$, for some $i \in \mathbb{N}$ such that $i \leq n$. Because $v = \%(\mathsf{00})\mathsf{0}^i$ and $\mathsf{11}$ is not a substring of $\mathsf{0}^i$, it follows that $v \notin X$.

- Suppose $v = \mathsf{000}^n\mathsf{1}$. Because $v = \%(\mathsf{00})(\mathsf{0}^n\mathsf{1})$ and $\mathsf{11}$ is not a substring of $\mathsf{0}^n\mathsf{1}$, it follows that $v \notin X$.

□

Proposition ES3.2.7
For all FAs N, if $L(N) = X$, then $|Q_N| \geq |Q_M|$.

Proof. Suppose, toward a contradiction, that it is not true that, for all FAs N, if $L(N) = X$, then $|Q_N| \geq |Q_M|$. Thus there is an FA N such that $L(N) = X$ and $|Q_N| < |Q_M|$. Because $|Q_M| = 3$, this means that $|Q_N| \leq 2$. Let n be two times the maximum length of the labels of N's transitions, and let $w = \mathsf{00}(\mathsf{0}^n)\mathsf{11}$. By Lemma ES3.2.6, $w \in X$ and $|w| = n + 4$. Because $w \in X = L(N)$, there is a labeled path *lp* such that *lp* is valid for N, the start state of *lp* is s_N, and the label of *lp* is w. By the definition of n, and since $|w| = n+4$, the labels of at least three transitions of *lp* must be non-%. Let lp' be the shortest initial part of *lp* that contains exactly two transitions with non-% labels, let v be the label of lp', and let q be the end state of lp'. Thus v is a prefix of w. By the definition of n, and because exactly two of the labels of lp' are non-%, we have that $2 \leq |v| \leq n \leq n + 3$. Furthermore, by Lemma ES3.2.5, we have that $\Delta(\{q\}, \%) \cap A_N \neq \emptyset$. Thus, there is a labeled path that is valid for N, starts at s_N, ends at an element of A_N, and is labeled by $v\% = v$, showing that $v \in L(N) = X$. But by Lemma ES3.2.6, it follows that $v \notin X$—contradiction. Thus, for all FAs N, if $L(N) = X$, then $|Q_N| \geq |Q_M|$. □