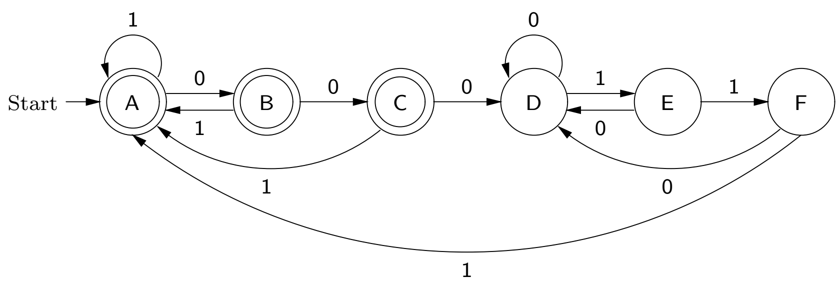
CIS 570 — Introduction to Formal Language Theory — Fall 2007

Exercise Set 4

Model Answers

Exercise 1

(a)



(b) Define the following languages:

$$
\begin{aligned}
Y &= \{\, w \in \{0,1\}^* \mid w \notin X \,\}, \\
A &= \{\, w \in \{0,1\}^* \mid w \in X \text{ and } 0 \text{ is not a suffix of } w \,\}, \\
B &= \{\, w \in \{0,1\}^* \mid w \in X \text{ and } 0, \text{ but not } 00, \text{ is a suffix of } w \,\}, \\
C &= \{\, w \in \{0,1\}^* \mid w \in X \text{ and } 00 \text{ is a suffix of } w \,\}, \\
D &= \{\, w \in \{0,1\}^* \mid w \notin X \text{ and } 0 \text{ is a suffix of } w \,\}, \\
E &= \{\, w \in \{0,1\}^* \mid w \notin X \text{ and } 1, \text{ but not } 11, \text{ is a suffix of } w \,\}, \\
F &= \{\, w \in \{0,1\}^* \mid w \notin X \text{ and } 11 \text{ is a suffix of } w \,\}.
\end{aligned}
$$

Lemma ES4.1.1

(1) $\% \in A$.

(2) $X\{1\} \subseteq X$.

(3) $A\{1\} \subseteq A$.

(4) $B\{1\} \subseteq A$.

(5) $C\{1\} \subseteq A$.

(6) $F\{1\} \subseteq A$.

(7) $A\{0\} \subseteq B$.

(8) $B\{0\} \subseteq C$.

(9) $C\{0\} \subseteq D$.

(10) $Y\{\mathtt{0}\} \subseteq Y$.

(11) $D\{\mathtt{0}\} \subseteq D$.

(12) $E\{\mathtt{0}\} \subseteq D$.

(13) $F\{\mathtt{0}\} \subseteq D$.

(14) $D\{\mathtt{1}\} \subseteq E$.

(15) $E\{\mathtt{1}\} \subseteq F$.

Proof.

(1) We must show that $\% \in A$. Because $\mathtt{0}$ is not a suffix of $\%$, it remains to show that $\% \in X$. Suppose that $x, y \in \{\mathtt{0},\mathtt{1}\}^*$ and $\% = x\mathtt{000}y$. We must show that $\mathtt{111}$ is a substring of y. But $\% = x\mathtt{000}y$ is impossible, and thus $\mathtt{111}$ is a substring of y.

(2) Suppose $w \in X\{\mathtt{1}\}$. We must show that $w \in X$. By the assumption, we have that $w = z\mathtt{1}$ for some $z \in X$. To show that $w \in X$, suppose $x, y \in \{\mathtt{0},\mathtt{1}\}^*$ and $w = x\mathtt{000}y$. We must show that $\mathtt{111}$ is a substring of y. Because $x\mathtt{000}y = w = z\mathtt{1}$, we have that $y = y'\mathtt{1}$ for some $y' \in \{\mathtt{0},\mathtt{1}\}^*$. Thus $x\mathtt{000}y'\mathtt{1} = x\mathtt{000}y = z\mathtt{1}$, so that $x\mathtt{000}y' = z$. Because $z \in X$, we have that $\mathtt{111}$ is a substring of y'. Thus $\mathtt{111}$ is a substring of $y'\mathtt{1} = y$.

(3) Suppose $w \in A\{\mathtt{1}\}$. We must show that $w \in A$. By the assumption, we have that $w = z\mathtt{1}$ for some $z \in A$. But $A \subseteq X$, and thus $w = z\mathtt{1} \in X\{\mathtt{1}\} \subseteq X$, by Part (2). And $\mathtt{0}$ is not a suffix of $z\mathtt{1} = w$, completing the proof that $w \in A$.

(4) Suppose $w \in B\{\mathtt{1}\}$. We must show that $w \in A$. By the assumption, we have that $w = z\mathtt{1}$ for some $z \in B$. But $B \subseteq X$, and thus $w = z\mathtt{1} \in X\{\mathtt{1}\} \subseteq X$, by Part (2). And $\mathtt{0}$ is not a suffix of $z\mathtt{1} = w$, completing the proof that $w \in A$.

(5) Suppose $w \in C\{\mathtt{1}\}$. We must show that $w \in A$. By the assumption, we have that $w = z\mathtt{1}$ for some $z \in C$. But $C \subseteq X$, and thus $w = z\mathtt{1} \in X\{\mathtt{1}\} \subseteq X$, by Part (2). And $\mathtt{0}$ is not a suffix of $z\mathtt{1} = w$, completing the proof that $w \in A$.

(6) Suppose $w \in F\{\mathtt{1}\}$. We must show that $w \in A$. By the assumption, we have that $w = z\mathtt{1}$ for some $z \in F$. Thus $\mathtt{0}$ is not a suffix of $z\mathtt{1} = w$. So it remains to show that $w \in X$. Suppose $x, y \in \{\mathtt{0},\mathtt{1}\}^*$ and $w = x\mathtt{000}y$. We must show that $\mathtt{111}$ is a substring of y. Because $z \in F$, we have that $\mathtt{11}$ is a suffix of z. Thus $z = z'\mathtt{11}$ for some $z' \in \{\mathtt{0},\mathtt{1}\}^*$, so that $w = z\mathtt{1} = z'\mathtt{111}$. Since $z'\mathtt{111} = w = x\mathtt{000}y$, we have that $\mathtt{111}$ is a suffix of y. Hence $\mathtt{111}$ is a substring of y.

(7) Suppose $w \in A\{\mathtt{0}\}$. We must show that $w \in B$. By the assumption, $w = z\mathtt{0}$ for some $z \in A$. We have that $\mathtt{0}$ is a suffix of $z\mathtt{0} = w$. Because $z \in A$, it follows that $\mathtt{0}$ is not a suffix of z. Thus $\mathtt{00}$ is not a suffix of $z\mathtt{0} = w$. So it remains to show that $w \in X$. Suppose $x, y \in \{\mathtt{0},\mathtt{1}\}^*$ and $w = x\mathtt{000}y$. We must show that $\mathtt{111}$ is a substring of y. We have that $x\mathtt{000}y = w = z\mathtt{0}$. Because $\mathtt{0}$ is not a suffix of z, there are two cases to consider.

- Suppose $z = \%$. Thus $x\mathtt{000}y = z\mathtt{0} = \mathtt{0}$—contradiction. Thus $\mathtt{111}$ is a substring of y.

- Suppose $z = z'1$ for some $z' \in \{0,1\}^*$. Thus $x000y = z0 = z'10$, so that $y = y'0$ for some $y' \in \{0,1\}^*$. Hence $x000y'0 = x000y = z'10$, so that $x000y' = z'1 = z$. Because $z \in A \subseteq X$, we have that 111 is a substring of y'. Thus 111 is a substring of $y'0 = y$.

(8) Suppose $w \in B\{0\}$. We must show that $w \in C$. By the assumption, $w = z0$ for some $z \in B$. Because $z \in B$, we have that 0, but not 00, is a suffix of z. Thus we have that 00 is a suffix of $z0 = w$. So, it remains to show that $w \in X$. Suppose $x, y \in \{0,1\}^*$ and $w = x000y$. We must show that 111 is a substring of y. We have that $x000y = w = z0$. Because 0, but not 00, is a suffix of z, there are two cases to consider.

- Suppose $z = 0$. Thus $x000y = z0 = 00$—contradiction. Thus 111 is a substring of y.
- Suppose $z = z'10$ for some $z' \in \{0,1\}^*$. Thus $x000y = z0 = z'100$, so that $y = y'100$ for some $y' \in \{0,1\}^*$. Hence $x000y'100 = x000y = z'100$, so that $x000y'10 = z'10 = z$. Because $z \in B \subseteq X$, we have that 111 is a substring of $y'10$. Thus 111 is a substring of $y'100 = y$.

(9) Suppose $w \in C\{0\}$. We must show that $w \in D$. By the assumption, $w = z0$ for some $z \in C$. Thus 0 is a suffix of $z0 = w$. So, it remains to show that $w \notin X$. Because $z \in C$, we have that 00 is a suffix of z. Thus $z = z'00$ for some $z' \in \{0,1\}^*$, so that $w = z0 = z'000 = z'000\%$. Because $w = z'000\%$ and 111 is not a substring of %, we have that $w \notin X$.

(10) Suppose $w \in Y\{0\}$. We must show that $w \in Y$. By the assumption, $w = z0$ for some $z \in Y$. Thus $z \notin X$, so that there are $x, y \in \{0,1\}^*$ such that $z = x000y$ and 111 is not a substring of y. Thus $w = z0 = x000y0$ and 111 is not a substring of $y0$, showing that $w \notin X$. Thus $w \in Y$.

(11) Suppose $w \in D\{0\}$. We must show that $w \in D$. By the assumption, $w = z0$ for some $z \in D$. Because $z \in D$, we have that $z \notin X$, so that $z \in Y$. Thus $w = z0 \in Y\{0\} \subseteq Y$, by Part (10), so that $w \notin X$. Furthermore, 0 is a suffix of $z0 = w$, completing the proof that $w \in D$.

(12) Suppose $w \in E\{0\}$. We must show that $w \in D$. By the assumption, $w = z0$ for some $z \in E$. Because $z \in E$, we have that $z \notin X$, so that $z \in Y$. Thus $w = z0 \in Y\{0\} \subseteq Y$, by Part (10), so that $w \notin X$. Furthermore, 0 is a suffix of $z0 = w$, completing the proof that $w \in D$.

(13) Suppose $w \in F\{0\}$. We must show that $w \in D$. By the assumption, $w = z0$ for some $z \in F$. Because $z \in F$, we have that $z \notin X$, so that $z \in Y$. Thus $w = z0 \in Y\{0\} \subseteq Y$, by Part (10), so that $w \notin X$. Furthermore, 0 is a suffix of $z0 = w$, completing the proof that $w \in D$.

(14) Suppose $w \in D\{1\}$. We must show that $w \in E$. By the assumption, we have that $w = z1$ for some $z \in D$. Thus $z \notin X$ and 0 is a suffix of z, so that $z = z'0$ for some $z' \in \{0,1\}^*$. Because $w = z1 = z'01$, we have that 1, but not 11, is a suffix of $z'01 = w$. So, it remains to show that $w \notin X$. Because $z \notin X$, there are $x, y \in \{0,1\}^*$ such that $z = x000y$ and 111 is not a substring of y. Thus $w = z1 = x000y1$. Suppose, toward a contradiction, that 111 is a substring of $y1$. Because 111 is not a substring of y, we have that 11 is a suffix of y. Thus, since $z'01 = w = x000y1$, we have that $0 = 1$—contradiction. Hence 111 is not a substring of $y1$. Because $w = x000y1$ and 111 is not a substring of $y1$, we have that $w \notin X$.

(15) Suppose $w \in E\{1\}$. We must show that $w \in F$. By the assumption, we have that $w = z1$ for some $z \in E$. Thus $z \notin X$ and 1, but not 11, is a suffix of z. Hence $z = z'1$ for some $z' \in \{0,1\}^*$, and 1 is not a suffix of z'. We have that 11 is a suffix of $z'11 = z1 = w$. So, it remains to show that $w \notin X$. Because $z \notin X$, there are $x, y \in \{0,1\}^*$ such that $z = x000y$ and 111 is not a substring of y. Hence $w = z1 = x000y1$. Suppose, toward a contradiction, that 111 is a substring of $y1$. Because 111 is not a substring of y, it follows that 11 is a suffix of y. Thus, since $z'11 = w = x000y1$, it follows that 1 is a suffix of z'—contradiction. Hence 111 is not a substring of $y1$. Because $w = x000y1$ and 111 is not a substring of $y1$, we have that $w \notin X$.

□

Lemma ES4.1.2
For all $w \in \{0,1\}^*$*:*

(A) if $\delta(\mathsf{A}, w) = \mathsf{A}$*, then* $w \in A$*.*

(B) if $\delta(\mathsf{A}, w) = \mathsf{B}$*, then* $w \in B$*.*

(C) if $\delta(\mathsf{A}, w) = \mathsf{C}$*, then* $w \in C$*.*

(D) if $\delta(\mathsf{A}, w) = \mathsf{D}$*, then* $w \in D$*.*

(E) if $\delta(\mathsf{A}, w) = \mathsf{E}$*, then* $w \in E$*.*

(F) if $\delta(\mathsf{A}, w) = \mathsf{F}$*, then* $w \in F$*.*

Proof. We proceed by left string induction.

(Basis Step) We must show that (A)–(F) hold, where % has been substituted for w. By Lemma ES4.1.1(1), we have that $\% \in A$, so that Part (A) holds. And Parts (B)–(F) hold vacuously, since $\delta(\mathsf{A}, \%) = \mathsf{A}$.

(Inductive Step) Suppose $a \in \{0,1\}$ and $w \in \{0,1\}^*$. Assume the inductive hypothesis, that Parts (A)–(F) hold. We must show that Parts (A)–(F) hold, where wa has been substituted for w. There are six parts to show.

(A) Suppose $\delta(\mathsf{A}, wa) = \mathsf{A}$. Since $\delta(\delta(\mathsf{A}, w), a) = \delta(\mathsf{A}, wa) = \mathsf{A}$, we have that $(\delta(\mathsf{A}, w), a, \mathsf{A}) \in T$. There are four cases to consider.

- Suppose $\delta(\mathsf{A}, w) = \mathsf{A}$ and $a = 1$. Part (A) of the inductive hypothesis tells us that $w \in A$. Thus $wa = w1 \in A\{1\} \subseteq A$, by Lemma ES4.1.1(3).
- Suppose $\delta(\mathsf{A}, w) = \mathsf{B}$ and $a = 1$. Part (B) of the inductive hypothesis tells us that $w \in B$. Thus $wa = w1 \in B\{1\} \subseteq A$, by Lemma ES4.1.1(4).
- Suppose $\delta(\mathsf{A}, w) = \mathsf{C}$ and $a = 1$. Part (C) of the inductive hypothesis tells us that $w \in C$. Thus $wa = w1 \in C\{1\} \subseteq A$, by Lemma ES4.1.1(5).
- Suppose $\delta(\mathsf{A}, w) = \mathsf{F}$ and $a = 1$. Part (F) of the inductive hypothesis tells us that $w \in F$. Thus $wa = w1 \in F\{1\} \subseteq A$, by Lemma ES4.1.1(6).

(B) Suppose $\delta(\mathsf{A}, wa) = \mathsf{B}$. Since) $\delta(\delta(\mathsf{A}, w), a) = \delta(\mathsf{A}, wa) = \mathsf{B}$, we have that $(\delta(\mathsf{A}, w), a, \mathsf{B}) \in T$. Thus $\delta(\mathsf{A}, w) = \mathsf{A}$ and $a = \mathsf{0}$. Part (A) of the inductive hypothesis tells us that $w \in A$. Thus $wa = w\mathsf{0} \in A\{\mathsf{0}\} \subseteq B$, by Lemma ES4.1.1(7).

(C) Suppose $\delta(\mathsf{A}, wa) = \mathsf{C}$. Since $\delta(\delta(\mathsf{A}, w), a) = \delta(\mathsf{A}, wa) = \mathsf{C}$, we have that $(\delta(\mathsf{A}, w), a, \mathsf{C}) \in T$. Thus $\delta(\mathsf{A}, w) = \mathsf{B}$ and $a = \mathsf{0}$. Part (B) of the inductive hypothesis tells us that $w \in B$. Thus $wa = w\mathsf{0} \in B\{\mathsf{0}\} \subseteq C$, by Lemma ES4.1.1(8).

(D) Suppose $\delta(\mathsf{A}, wa) = \mathsf{D}$. Since $\delta(\delta(\mathsf{A}, w), a) = \delta(\mathsf{A}, wa) = \mathsf{D}$, we have that $(\delta(\mathsf{A}, w), a, \mathsf{D}) \in T$. There are four cases to consider.

- Suppose $\delta(\mathsf{A}, w) = \mathsf{C}$ and $a = \mathsf{0}$. Part (C) of the inductive hypothesis tells us that $w \in C$. Thus $wa = w\mathsf{0} \in C\{\mathsf{0}\} \subseteq D$, by Lemma ES4.1.1(9).
- Suppose $\delta(\mathsf{A}, w) = \mathsf{D}$ and $a = \mathsf{0}$. Part (D) of the inductive hypothesis tells us that $w \in D$. Thus $wa = w\mathsf{0} \in D\{\mathsf{0}\} \subseteq D$, by Lemma ES4.1.1(11).
- Suppose $\delta(\mathsf{A}, w) = \mathsf{E}$ and $a = \mathsf{0}$. Part (E) of the inductive hypothesis tells us that $w \in E$. Thus $wa = w\mathsf{0} \in E\{\mathsf{0}\} \subseteq D$, by Lemma ES4.1.1(12).
- Suppose $\delta(\mathsf{A}, w) = \mathsf{F}$ and $a = \mathsf{0}$. Part (F) of the inductive hypothesis tells us that $w \in F$. Thus $wa = w\mathsf{0} \in F\{\mathsf{0}\} \subseteq D$, by Lemma ES4.1.1(13).

(E) Suppose $\delta(\mathsf{A}, wa) = \mathsf{E}$. Since $\delta(\delta(\mathsf{A}, w), a) = \delta(\mathsf{A}, wa) = \mathsf{E}$, we have that $(\delta(\mathsf{A}, w), a, \mathsf{E}) \in T$. Thus $\delta(\mathsf{A}, w) = \mathsf{D}$ and $a = \mathsf{1}$. Part (D) of the inductive hypothesis tells us that $w \in D$. Thus $wa = w\mathsf{1} \in D\{\mathsf{1}\} \subseteq E$, by Lemma ES4.1.1(14).

(F) Suppose $\delta(\mathsf{A}, wa) = \mathsf{F}$. Since $\delta(\delta(\mathsf{A}, w), a) = \delta(\mathsf{A}, wa) = \mathsf{F}$, we have that $(\delta(\mathsf{A}, w), a, \mathsf{F}) \in T$. Thus $\delta(\mathsf{A}, w) = \mathsf{E}$ and $a = \mathsf{1}$. Part (E) of the inductive hypothesis tells us that $w \in E$. Thus $wa = w\mathsf{1} \in E\{\mathsf{1}\} \subseteq F$, by Lemma ES4.1.1(15).

□

Now, we use the preceding lemma to show that $L(M) = X$.

($L(M) \subseteq X$) Suppose $w \in L(M)$. Then $w \in \{\mathsf{0}, \mathsf{1}\}^*$ and $\delta(\mathsf{A}, w) \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$. Thus, by Parts (A)–(C) of Lemma ES4.1.2, we have that $w \in A$ or $w \in B$ or $w \in C$. But $A \subseteq X$, $B \subseteq X$ and $C \subseteq X$, so that $w \in X$.

($X \subseteq L(M)$) Suppose $w \in X$. Since $X \subseteq \{\mathsf{0}, \mathsf{1}\}^*$, we have that $w \in \{\mathsf{0}, \mathsf{1}\}^*$. Suppose, toward a contradiction, that $w \notin L(M)$. Thus $\delta(\mathsf{A}, w) \in \{\mathsf{D}, \mathsf{E}, \mathsf{F}\}$. Thus, by Parts (D)–(F) of Lemma ES4.1.2, we have that $w \in D$ or $w \in E$ or $w \in F$. Thus $w \notin X$—contradiction. Thus $w \in L(M)$.

Exercise 2

(a) The regular expression α is $((\mathsf{01})^*)^*$. To see that our answer is correct, we put the description

```
{states}
A, B, C, D
{start state}
A
{accepting states}
```

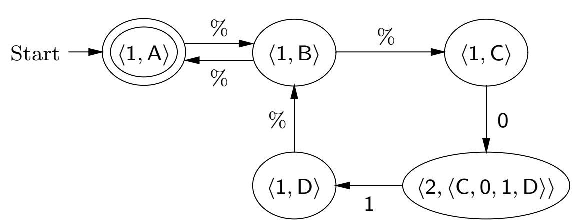
```
A
{transitions}
A, % -> B; B, % -> A | C; C, 01 -> D; D, % -> B
```

of the FA M_1 in the file `es4-ex2-fa`. We then load α and M_1 into Forlan, convert α to an FA, and check that it is isomorphic to M_1, as follows:

```
- val reg = Reg.input "";
@ ((01)*)*
@ .
val reg = - : reg
- val fa1 = FA.input "es4-ex2-fa";
val fa1 = - : fa
- FA.isomorphic(regToFA reg, fa1);
val it = true : bool
```

(b) M_2 is formed from M_1 by the following process. First, the state A of M_1 is turned into the state $\langle 1, \mathsf{A}\rangle$ of M_2. $\langle 1, \mathsf{A}\rangle$ is the start state of M_2, since A is the start state of M_1; it is an accepting state, since A is an accepting state. Next, the state B of M_1 is turned into the state $\langle 1, \mathsf{B}\rangle$ of M_2; $\langle 1, \mathsf{B}\rangle$ is a non-accepting state, since B is a non-accepting state. Similarly, the states C and D of M_1 are turned into the non-accepting states $\langle 1, \mathsf{C}\rangle$ and $\langle 1, \mathsf{D}\rangle$ of M_2. Next, the transitions $(\mathsf{A}, \%, \mathsf{B})$, $(\mathsf{B}, \%, \mathsf{A})$, $(\mathsf{B}, \%, \mathsf{C})$ and $(\mathsf{D}, \%, \mathsf{B})$ of M_1 are turned into the transitions $(\langle 1, \mathsf{A}\rangle, \%, \langle 1, \mathsf{B}\rangle)$, $(\langle 1, \mathsf{B}\rangle, \%, \langle 1, \mathsf{A}\rangle)$, $(\langle 1, \mathsf{B}\rangle, \%, \langle 1, \mathsf{C}\rangle)$ and $(\langle 1, \mathsf{D}\rangle, \%, \langle 1, \mathsf{B}\rangle)$ of M_2, respectively. Finally, the transition $(\mathsf{C}, \mathsf{01}, \mathsf{D})$ of M_1 is split into the M_2 transitions $(\langle 1, \mathsf{C}\rangle, \mathsf{0}, \langle 2, \langle \mathsf{C}, 0, 1, \mathsf{D}\rangle\rangle)$ and $(\langle 2, \langle \mathsf{C}, 0, 1, \mathsf{D}\rangle\rangle, \mathsf{1}, \langle 1, \mathsf{D}\rangle)$, where $\langle 2, \langle \mathsf{C}, 0, 1, \mathsf{D}\rangle\rangle$ is a new, non-accepting state of M_2.

Here is a drawing of M_2:



Continuing our Forlan session, We convert the FA M_1 into an EFA, as follows:

```
- val efa2 = faToEFA fa1;
val efa2 = - : efa
- EFA.output("", efa2);
{states}
<1,A>, <1,B>, <1,C>, <1,D>, <2,<C,0,1,D>>
{start state}
<1,A>
{accepting states}
<1,A>
{transitions}
<1,A>, % -> <1,B>; <1,B>, % -> <1,A> | <1,C>; <1,C>, 0 -> <2,<C,0,1,D>>;
<1,D>, % -> <1,B>; <2,<C,0,1,D>>, 1 -> <1,D>
```

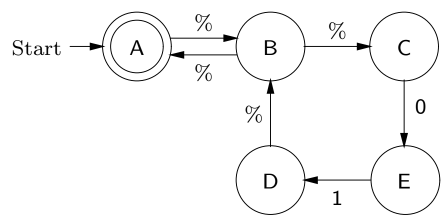
```
val it = () : unit
```

It is easy to check that M_2 is the outputted EFA.

(c) Continuing with our Forlan session, we rename the states of M_2, producing an EFA M_3, as follows:

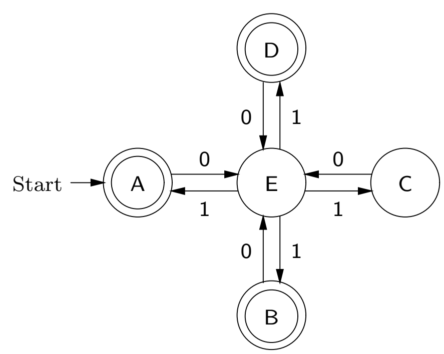
```
- val efa3 = EFA.renameStatesCanonically efa2;
val efa3 = - : efa
- EFA.output("", efa3);
{states}
A, B, C, D, E
{start state}
A
{accepting states}
A
{transitions}
A, % -> B; B, % -> A | C; C, 0 -> E; D, % -> B; E, 1 -> D
val it = () : unit
```

Here is a drawing of M_3:



(d) The NFA M_4 is formed from M_3 by the following process. Its states and start state are the same as those of M_3. The accepting states of M_4 are the elements of $\mathbf{emptyCloseBackwards}(A_{M_3}) = \mathbf{emptyCloseBackwards}(\{\mathsf{A}\}) = \{\mathsf{A}, \mathsf{B}, \mathsf{D}\}$. The transitions of M_4 are formed by processing the non-% transitions of M_3. The transition $(\mathsf{C}, 0, \mathsf{E})$ of M_3 is turned into the transitions $(\mathsf{A}, 0, \mathsf{E})$, $(\mathsf{B}, 0, \mathsf{E})$, $(\mathsf{C}, 0, \mathsf{E})$ and $(\mathsf{D}, 0, \mathsf{E})$, since $\mathbf{emptyCloseBackwards}(\{\mathsf{C}\}) = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}$ and $\mathbf{emptyClose}(\{\mathsf{E}\}) = \{\mathsf{E}\}$. The transition $(\mathsf{E}, 1, \mathsf{D})$ of M_3 is turned into the transitions $(\mathsf{E}, 1, \mathsf{A})$, $(\mathsf{E}, 1, \mathsf{B})$, $(\mathsf{E}, 1, \mathsf{C})$ and $(\mathsf{E}, 1, \mathsf{D})$, since $\mathbf{emptyCloseBackwards}(\{\mathsf{E}\}) = \{\mathsf{E}\}$ and $\mathbf{emptyClose}(\{\mathsf{D}\}) = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}$.

Here is a drawing of M_4:



Continuing our Forlan session, we convert the EFA M_3 into an NFA, and output the result, as follows.

```
- val nfa4 = efaToNFA efa3;
val nfa4 = - : nfa
- NFA.output("", nfa4);
{states}
A, B, C, D, E
{start state}
A
{accepting states}
A, B, D
{transitions}
A, 0 -> E; B, 0 -> E; C, 0 -> E; D, 0 -> E; E, 1 -> A | B | C | D
val it = () : unit
```

It is easy to check that M_4 is the outputted NFA.

(e) We form the DFA M_5 from the NFA M_4, by the following process, which involves the construction of a set X of sets of states of M_4. First, $\{\mathsf{A}\}$ is added to X and $\langle\mathsf{A}\rangle$ is made the start state of M_5, since A is M_4's start state. Since A is an accepting state of M_4, $\langle\mathsf{A}\rangle$ is an accepting state of M_5.

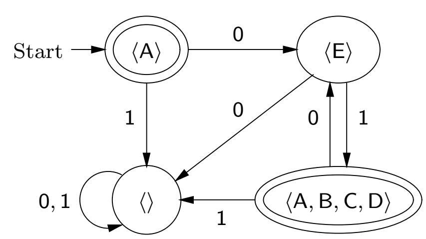
Since $\{\mathsf{A}\} \in X$ and $\Delta(\{\mathsf{A}\}, 0) = \{\mathsf{E}\}$, we add $\{\mathsf{E}\}$ to X, $\langle\mathsf{E}\rangle$ to Q_{M_5} and $(\langle\mathsf{A}\rangle, 0, \langle\mathsf{E}\rangle)$ to T_{M_5}. Since $\{\mathsf{A}\} \in X$ and $\Delta(\{\mathsf{A}\}, 1) = \emptyset$, we add $\emptyset$ to X, $\langle\rangle$ to Q_{M_5} and $(\langle\mathsf{A}\rangle, 1, \langle\rangle)$ to T_{M_5}.

Since $\{\mathsf{E}\} \in X$ and $\Delta(\{\mathsf{E}\}, 0) = \emptyset$, and $\emptyset$ is already in X, we add $(\langle\mathsf{E}\rangle, 0, \langle\rangle)$ to T_{M_5}. Since $\{\mathsf{E}\} \in X$ and $\Delta(\{\mathsf{E}\}, 1) = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}$, we add $\{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}$ to X, $\langle\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\rangle$ to Q_{M_5} and $(\langle\mathsf{E}\rangle, 1, \langle\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\rangle)$ to T_{M_5}. Since A is an accepting state of M_4, $\langle\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\rangle$ is an accepting state of M_5.

Since $\emptyset \in X$ and $\Delta(\emptyset, 0) = \emptyset$, we add $(\langle\rangle, 0, \langle\rangle)$ to T_{M_5}. Since $\emptyset \in X$ and $\Delta(\emptyset, 1) = \emptyset$, we add $(\langle\rangle, 1, \langle\rangle)$ to T_{M_5}.

Since $\{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\} \in X$ and $\Delta(\{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}, 0) = \{\mathsf{E}\} \cup \{\mathsf{E}\} \cup \{\mathsf{E}\} \cup \{\mathsf{E}\} = \{\mathsf{E}\}$, and $\{\mathsf{E}\}$ is already in X, we add $(\langle\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\rangle, 0, \langle\mathsf{E}\rangle)$ to T_{M_5}. Since $\{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\} \in X$ and $\Delta(\{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}, 1) = \emptyset \cup \emptyset \cup \emptyset \cup \emptyset = \emptyset$, and $\emptyset$ is already in X, we add $(\langle\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\rangle, 1, \langle\rangle)$ to T_{M_5}.

Here is a drawing of M_5:



Continuing our Forlan session, we convert our NFA M_4 into a DFA, as follows:

```
- val dfa5 = nfaToDFA nfa4;
val dfa5 = - : dfa
- DFA.output("", dfa5);
{states}
<>, <A>, <E>, <A,B,C,D>
```

```
{start state}
<A>
{accepting states}
<A>, <A,B,C,D>
{transitions}
<>, 0 -> <>; <>, 1 -> <>; <A>, 0 -> <E>; <A>, 1 -> <>; <E>, 0 -> <>;
<E>, 1 -> <A,B,C,D>; <A,B,C,D>, 0 -> <E>; <A,B,C,D>, 1 -> <>
val it = () : unit
```

It is easy to check that M_5 is the outputted DFA.

(f) Continuing our Forlan session, we rename the states of M_5, producing a DFA M_6, as follows:

```
- val dfa6 = DFA.renameStatesCanonically dfa5;
val dfa6 = - : dfa
- DFA.output("", dfa6);
{states}
A, B, C, D
{start state}
B
{accepting states}
B, D
{transitions}
A, 0 -> A; A, 1 -> A; B, 0 -> C; B, 1 -> A; C, 0 -> A; C, 1 -> D; D, 0 -> C;
D, 1 -> A
val it = () : unit
```

Here is a drawing of M_6: