CS 591 S2—Formal Language Theory: Integrating Experimentation and Proof—Fall 2018

Problem Set 1

Model Answers

Problem 1

To show that $\emptyset \to X = \{\emptyset\}$, we show that each side is a subset of the other.

Suppose $f \in \emptyset \to X$, so that f is a function, $\mathbf{domain}\, f = \emptyset$ and $\mathbf{range}\, f \subseteq X$. Because $\mathbf{domain}\, f = \emptyset$, we have that $f = \emptyset$. Thus $f \in \{\emptyset\}$.

Suppose $f \in \{\emptyset\}$, so that $f = \emptyset$. Then f is a function, $\mathbf{domain}\, f = \emptyset$ and $\mathbf{range}\, f = \emptyset \subseteq X$. Thus $f \in \emptyset \to X$.

To show that $X \to \emptyset = \emptyset$, we show that each side is a subset of the other.

Suppose $f \in X \to \emptyset$, so that f is a function, $\mathbf{domain}\, f = X$ and $\mathbf{range}\, f \subseteq \emptyset$. Consequently $\mathbf{range}\, f = \emptyset$. Because $x \in X = \mathbf{domain}\, f$, there is a y such that $(x, y) \in f$. But then $y \in \mathbf{range}\, f = \emptyset$—contradiction. Thus we can conclude anything, including that $f \in \emptyset$.

Suppose $f \in \emptyset$. Because this is a contradiction, we can conclude that $f \in X \to \emptyset$.

To show that $\{x\} \to X = \{\, \{(x, y)\} \mid y \in X \,\}$, we show that each side is a subset of the other.

Suppose $f \in \{x\} \to X$. Thus f is a function, $\mathbf{domain}\, f = \{x\}$ and $\mathbf{range}\, f \subseteq X$. Consequently, f is a relation including a pair of the form (x, y), for some y. Because $\mathbf{domain}\, f = \{x\}$, there are no elements of f whose left sides are not x. And because f is a function, there are no other pairs in f whose left sides are x. Thus $f = \{(x, y)\}$, so that $f \in \{\, \{(x, y)\} \mid y \in X \,\}$.

Suppose $f \in \{\, \{(x, y)\} \mid y \in X \,\}$. Thus $f = \{(x, y)\}$, for some $y \in X$. Hence f is a function, $\mathbf{domain}\, f = \{x\}$ and $\mathbf{range}\, f = \{y\} \subseteq X$. Thus $f \in \{x\} \to X$.

To show that $X \to \{x\} = \{\{\, (y, x) \mid y \in X \,\}\}$, we show that each side is a subset of the other.

Suppose $f \in X \to \{x\}$. Thus f is a function, $\mathbf{domain}\, f = X$ and $\mathbf{range}\, f \subseteq \{x\}$. To show that $f = \{\, (y, x) \mid y \in X \,\}$, we show that each side is a subset of the other.

- Suppose $p \in f$. From our assumptions, we know that $p = (y, x)$ for some $y \in X$. Thus $p \in \{\, (y, x) \mid y \in X \,\}$.
- Suppose $p \in \{\, (y, x) \mid y \in X \,\}$, so that $p = (y, x)$ for some $y \in X$. Because f is a function and $y \in X = \mathbf{domain}\, f$, we have that $(y, x') \in f$ for some x'. But $\mathbf{range}\, f \subseteq \{x\}$, and thus $x' = x$. Hence $p = (y, x) = (y, x') \in f$.

Because $f = \{\, (y, x) \mid y \in X \,\}$, we can conclude that $f \in \{\{\, (y, x) \mid y \in X \,\}\}$.

Suppose $f \in \{\{\, (y, x) \mid y \in X \,\}\}$, so that $f = \{\, (y, x) \mid y \in X \,\}$. Thus f is a relation, $\mathbf{domain}\, f \subseteq X$ and $\mathbf{range}\, f \subseteq \{x\}$. Furthermore, for all $y \in X$, $(y, x) \in f$, so that $y \in \mathbf{domain}\, f$. Thus $\mathbf{domain}\, f = X$. Because $\mathbf{range}\, f \subseteq \{x\}$, f must be a function. Summarizing, we have that f is a function, $\mathbf{domain}\, f = X$ and $\mathbf{range}\, f \subseteq \{x\}$, showing that $f \in X \to \{x\}$.

Problem 2

We use mathematical induction to show that, for all $n \in \mathbb{N}$, if $n \geq 4$, then $2^n < n!$.

(Basis Step) We must show that, if $0 \geq 4$, then $2^0 < 0!$. Suppose $0 \geq 4$. But this is a contradiction, and thus we can conclude that $2^0 < 0!$.

(Inductive Step) Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis:

$$\text{if } n \geq 4, \text{ then } 2^n < n!.$$

We must show that,

$$\text{if } n+1 \geq 4, \text{ then } 2^{n+1} < (n+1)!.$$

Suppose $n+1 \geq 4$. We must show that $2^{n+1} < (n+1)!$. Since $n+1 \geq 4$, we have that $n \geq 3$. There are two cases to consider.

- Suppose $n = 3$. Then
$$2^{n+1} = 2^4 = 16 < 24 = 4! = (n+1)!.$$
- Suppose $n \geq 4$. By the inductive hypothesis, we have that $2^n < n!$. Furthermore, $2 < n+1$, so that
$$2^{n+1} = 2 \cdot 2^n < 2 \cdot n! < (n+1) \cdot n! = (n+1)!.$$

Problem 3

First, note a fact that we call (3) below: for all $n \in \mathbb{N}$,

$$f^2(n) = f^{1+1}(n) = f(f^1(n)) = f(f\,n),$$

where the last step is by (1).

We use strong induction to show that, for all $n \in \mathbb{N}$, there is an $l \in \mathbb{N}$ such that $f^l(n) = 0$. Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: for all $m \in \mathbb{N}$, if $m < n$, then there is an $l \in \mathbb{N}$ such that $f^l(m) = 0$. We must show that there is an $l \in \mathbb{N}$ such that $f^l(n) = 0$. There are four cases to consider.

($n = 0$) We have that $f^0(n) = n = 0$.

($n = 1$) We have that $f^1(n) = f^1(1) = f\,1 = 0$, by (1) and the definition of f.

($n > 1$ and n is even) Since n is even, we have that $n = 2i$ for some $i \in \mathbb{N}$. And, because $2i = n > 1$, we can conclude that $i \geq 1$. Hence $i < i + i$, with the consequence that

$$\frac{n}{2} = \frac{2i}{2} = i < i + i = 2i = n.$$

Hence $n/2 < n$ and $n/2 = i \in \mathbb{N}$. Thus, by the inductive hypothesis, it follows that there is an $l \in \mathbb{N}$ such that $f^l(n/2) = 0$. Hence $1 + l \in \mathbb{N}$ and

$$\begin{aligned}
f^{1+l}(n) &= f^l(f^1(n)) && \text{(by (2))} \\
&= f^l(f\,n)) && \text{(by (1))} \\
&= f^l(n/2) && \text{(definition of } f(n)\text{, since } n \text{ is even)} \\
&= 0.
\end{aligned}$$

($n > 1$ and n is odd) Since n is odd, we have that $n = 2i + 1$ for some $i \in \mathbb{N}$. And, because $2i + 1 = n > 1$, we can conclude that $i \geq 1$. Hence $i + 1 < i + i + 1$, with the consequence that

$$\frac{n+1}{2} = \frac{(2i+1)+1}{2} = \frac{2i+2}{2} = \frac{2(i+1)}{2} = i + 1 < i + i + 1 = 2i + 1 = n.$$

Hence $(n+1)/2 < n$ and $(n+1)/2 = i + 1 \in \mathbb{N}$. Thus, by the inductive hypothesis, there is an $l \in \mathbb{N}$ such that $f^l((n+1)/2) = 0$. Hence $2 + l \in \mathbb{N}$ and

$$\begin{aligned}
f^{2+l}(n) &= f^l(f^2(n)) && \text{(by (2))} \\
&= f^l(f(f\,n)) && \text{(by (3))} \\
&= f^l(f(n+1)) && \text{(definition of } f(n)\text{, since } n > 1 \text{ and } n \text{ is odd)} \\
&= f^l((n+1)/2) && \text{(definition of } f(n+1)\text{, since } n+1 \text{ is even)} \\
&= 0.
\end{aligned}$$

Problem 4

Let our property $P(x)$, for $x \in A$, be the statement

$$0 \neq 0.$$

Thus $P(x)$ is false, no matter whether $x = 0$ or $x = 1$. Note that $P(x)$ doesn't even depend on x. Consequently, we have that $P(0)$ holds iff $P(1)$ holds.

We use well-founded induction on R to show that, for all $x \in A$, $P(x)$. Suppose $x \in A$, and assume the inductive hypothesis: for all $y \in A$, if $y\ R\ x$, then $P(y)$. We must show that $P(x)$. Because $x \in A = \{0, 1\}$, there are two cases to consider.

- Suppose $x = 0$. Because $1 \in A$ and $1\ R\ 0 = x$, the inductive hypothesis tells us that $P(1)$. But this is equivalent to $P(0)$, i.e., $P(x)$.

- Suppose $x = 1$. Because $0 \in A$ and $0\ R\ 1 = x$, the inductive hypothesis tells us that $P(0)$. But this is equivalent to $P(1)$, i.e., $P(x)$.

Problem 5

To disprove the statement, we will show there is a set X and a nonempty set of X-closed sets $\mathcal{W}$ such that $\bigcup \mathcal{W}$ is not X-closed.

Let $X = \{0\}$ and $\mathcal{W} = \{\mathbf{Tree}\,\{0,1\}, \mathbf{Tree}\,\{0,2\}\}$. By definition, $\mathbf{Tree}\,\{0,1\}$ is $\{0,1\}$-closed. But $\{0\} \subseteq \{0,1\}$, and thus $\mathbf{Tree}\,\{0,1\}$ is also $\{0\}$-closed. By definition, $\mathbf{Tree}\,\{0,2\}$ is $\{0,2\}$-closed. But $\{0\} \subseteq \{0,2\}$, and thus $\mathbf{Tree}\,\{0,2\}$ is also $\{0\}$-closed. Thus $\mathcal{W}$ is indeed a nonempty set of $\{0\}$-closed sets, i.e., X-closed sets. Furthermore $\bigcup \mathcal{W} = \mathbf{Tree}\,\{0,1\} \cup \mathbf{Tree}\,\{0,2\}$.

Suppose, toward a contradiction, that $\bigcup \mathcal{W}$ is X-closed. Because $\mathbf{Tree}\,\{0,1\}$ is $\{0,1\}$-closed, $1 \in \{0,1\}$ and $[\,] \in \mathbf{List}(\mathbf{Tree}\,\{0,1\})$, we have that $(1,[\,]) \in \mathbf{Tree}\,\{0,1\} \subseteq \bigcup \mathcal{W}$. Because $\mathbf{Tree}\,\{0,2\}$ is $\{0,2\}$-closed, $2 \in \{0,2\}$ and $[\,] \in \mathbf{List}(\mathbf{Tree}\,\{0,2\})$, we have that $(2,[\,]) \in \mathbf{Tree}\,\{0,2\} \subseteq \bigcup \mathcal{W}$. Let $trs = [(1,[\,]),(2,[\,])]$. Then $trs \in \mathbf{List}\bigcup \mathcal{W}$. But $\bigcup \mathcal{W}$ is X-closed and $0 \in X$, and thus $(0, trs) \in \bigcup \mathcal{W}$. Because $\bigcup \mathcal{W} = \mathbf{Tree}\,\{0,1\} \cup \mathbf{Tree}\,\{0,2\}$, there are two cases to consider.

- Suppose $(0, trs) \in \mathbf{Tree}\,\{0,1\}$. By Proposition 1.3.4, we have that $trs \in \mathbf{List}(\mathbf{Tree}\,\{0,1\})$. Consequently, $(2, [\,]) \in \mathbf{Tree}\,\{0,1\}$. But then Proposition 1.3.4 tells us that $2 \in \{0,1\}$—contradiction.
- Suppose $(0, trs) \in \mathbf{Tree}\,\{0,2\}$. By Proposition 1.3.4, we have that $trs \in \mathbf{List}(\mathbf{Tree}\,\{0,2\})$. Consequently, $(1, [\,]) \in \mathbf{Tree}\,\{0,2\}$. But then Proposition 1.3.4 tells us that $1 \in \{0,2\}$—contradiction.

Because we achieved a contradiction in both cases, we have an overall contradiction. Thus we can conclude that $\bigcup \mathcal{W}$ is not X-closed.