CS 591 S2—Formal Language Theory: Integrating Experimentation and Proof—Fall 2019

Problem Set 1

Model Answers

Problem 1

(a) Suppose A, B and C are sets. We must show that

$$A - (B \cup C) = (A - B) - C.$$

It will suffice to show that

$$A - (B \cup C) \subseteq (A - B) - C \subseteq A - (B \cup C).$$

$(A - (B \cup C) \subseteq (A - B) - C)$ Suppose $w \in A - (B \cup C)$. We must show that $w \in (A - B) - C$. By the assumption, we have that $w \in A$ and $w \notin (B \cup C)$.

Suppose, toward a contradiction, that $w \in B$. Then $w \in B \cup C$—contradiction. Thus $w \notin B$.

Suppose, toward a contradiction, that $w \in C$. Then $w \in B \cup C$—contradiction. Thus $w \notin C$.

Because $w \in A$ and $w \notin B$, we have that $w \in A - B$. Then, since $w \notin C$, it follows that $w \in (A - B) - C$.

$((A - B) - C \subseteq A - (B \cup C))$ Suppose $w \in (A - B) - C$. We must show that $w \in A - (B \cup C)$. By the assumption, we have that $w \in A - B$ and $w \notin C$. Hence $w \in A$ and $w \notin B$.

Suppose, toward a contradiction, that $w \in B \cup C$. There are two cases to consider.

- Suppose $w \in B$. But $w \notin B$—contradiction.
- Suppose $w \in C$. But $w \notin C$—contradiction.

Since we obtained a contradiction in both cases, we have an overall contradiction. Thus $w \notin B \cup C$.

Because $w \in A$ and $w \notin B \cup C$, we have that $w \in A - (B \cup C)$.

(b) Suppose A, B and C are sets. We must show that

$$A - (B \cap C) = (A - B) \cup (A - C).$$

It will suffice to show that

$$A - (B \cap C) \subseteq (A - B) \cup (A - C) \subseteq A - (B \cap C).$$

$(A - (B \cap C) \subseteq (A - B) \cup (A - C))$ Suppose $w \in A - (B \cap C)$. We must show that $w \in (A - B) \cup (A - C)$. By the assumption, we have that $w \in A$ and $w \notin B \cap C$. There are two cases to consider.

- Suppose $w \in B$. Suppose, toward a contradiction, that $w \in C$. Thus $w \in B \cap C$—contradiction. Thus $w \notin C$. And $w \in A$, and thus $w \in A - C \subseteq (A - B) \cup (A - C)$.

- Suppose $w \notin B$. Because $w \in A$, it follows that $w \in A - B \subseteq (A - B) \cup (A - C)$.

$((A - B) \cup (A - C) \subseteq A - (B \cap C))$ Suppose $w \in (A - B) \cup (A - C)$. We must show that $w \in A - (B \cap C)$. By the assumption, there are two cases to consider.

- Suppose $w \in A - B$. Hence $w \in A$ and $w \notin B$. Suppose, toward a contradiction, that $w \in B \cap C$. Thus $w \in B$—contradiction. Hence $w \notin B \cap C$, so that $w \in A - (B \cap C)$.
- Suppose $w \in A - C$. Hence $w \in A$ and $w \notin C$. Suppose, toward a contradiction, that $w \in B \cap C$. Thus $w \in C$—contradiction. Hence $w \notin B \cap C$, so that $w \in A - (B \cap C)$.

Problem 2

We proceed by strong induction. Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: for all $m \in \mathbb{N}$, if $m < n$, then,

$$\text{if } m \geq 18, \text{ then there are } i, j \in \mathbb{N} \text{ such that } m = 4i + 7j.$$

We must show that,

$$\text{if } n \geq 18, \text{ then there are } i, j \in \mathbb{N} \text{ such that } n = 4i + 7j.$$

Suppose $n \geq 18$. We must show that there are $i, j \in \mathbb{N}$ such that $n = 4i + 7j$. There are five cases to consider.

- Suppose $n = 18$. Then $n = 18 = 4 * 1 + 7 * 2$ and $1, 2 \in \mathbb{N}$.
- Suppose $n = 19$. Then $n = 19 = 4 * 3 + 7 * 1$ and $3, 1 \in \mathbb{N}$.
- Suppose $n = 20$. Then $n = 20 = 4 * 5 + 7 * 0$ and $5, 0 \in \mathbb{N}$.
- Suppose $n = 21$. Then $n = 21 = 4 * 0 + 7 * 3$ and $0, 3 \in \mathbb{N}$.
- Suppose $n \geq 22$. Thus $18 \leq n - 4 < n$. Because $n - 4 < n$, the inductive hypothesis tells us that

  $$\text{if } n - 4 \geq 18, \text{ then there are } i, j \in \mathbb{N} \text{ such that } n - 4 = 4i + 7j.$$

  But $n - 4 \geq 18$, and thus $n - 4 = 4i + 7j$ for some $i, j \in \mathbb{N}$. Hence

  $$n = (n - 4) + 4 = 4i + 7j + 4 = 4(i + 1) + 7j,$$

  and $i + 1, j \in \mathbb{N}$.

Problem 3

Define a function $f \in \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by: for all $n \in \mathbb{N}$, $f\,n = (n, 0)$. To see that f is injective, suppose $n, m \in \mathbb{N}$ and $f\,n = f\,m$. Then $(n, 0) = f\,n = f\,m = (m, 0)$, so that $n = m$, as required. Thus $\mathbb{N} \preceq \mathbb{N} \times \mathbb{N}$. By the Schröder-Bernstein Theorem, it remains to show that $\mathbb{N} \times \mathbb{N} \preceq \mathbb{N}$.

Let $X = \{\, n \in \mathbb{N} \mid n \text{ is prime} \,\}$ and $Y = \{\, ns \in \mathbf{List}\, X \mid \text{for all } 1 \leq i < j \leq |ns|, ns\, i \leq ns\, j \,\}$. Define a function $g \in Y \to \mathbb{N}$ by: for all $ns \in Y$, $g\, ns$ is the product of ns. To see that g is injective, suppose $ns, ms \in Y$ and $g\, ns = g\, ms$. By the Fundamental Theorem of Arithmetic, we have that $ns = ms$. Thus g is injective, so that $Y \preceq \mathbb{N}$.

We define a function $\mathbf{rep} \in \mathbb{N} \times \mathbb{N} \to \mathbf{List}\, \mathbb{N}$ ("rep" stands for "repeat") by recursion on its second argument:

$$\mathbf{rep}(n, 0) = [\,], \text{ for all } n \in \mathbb{N}, \text{ and}$$
$$\mathbf{rep}(n, m+1) = [n] \mathbin{@} \mathbf{rep}(n, m), \text{ for all } n, m \in \mathbb{N}.$$

Define a function $h \in \mathbb{N} \times \mathbb{N} \to Y$ by: for all $n, m \in \mathbb{N}$, $h(n, m) = \mathbf{rep}(2, n) \mathbin{@} \mathbf{rep}(3, m)$. Because $2, 3 \in X$ and $2 \leq 3$, it follows that h is well-defined. To see that h is injective, suppose $n, n', m, m' \in \mathbb{N}$ and $h(n, m) = h(n', m')$. Thus

$$\mathbf{rep}(2, n) \mathbin{@} \mathbf{rep}(3, m) = h(n, m) = h(n', m') = \mathbf{rep}(2, n') \mathbin{@} \mathbf{rep}(3, m').$$

Since $\mathbf{rep}(2, n)$ and $\mathbf{rep}(2, n')$ consist entirely of 2's, whereas $\mathbf{rep}(3, m)$ and $\mathbf{rep}(3, m')$ have no 2's, it follows that $n = n'$, so that $m = m'$. Thus h is injective, so that $\mathbb{N} \times \mathbb{N} \preceq Y$.

Finally, because $\mathbb{N} \times \mathbb{N} \preceq Y \preceq \mathbb{N}$, we have that $\mathbb{N} \times \mathbb{N} \preceq \mathbb{N}$, completing the proof.

Problem 4

Because R is a relation on A, but is not well founded, it is not the case that every nonempty subset of A has an R-minimal element. Thus, there exists a nonempty subset X of A, such that X does not have an R-minimal element. In other words, there does not exist an $x \in X$ such that there does not exist a $y \in X$ such that $y \mathrel{R} x$. Hence (†): for all $x \in X$, there exists a $y \in X$, such that $y \mathrel{R} x$.

First, we use well-founded induction on R to show that, for all $x \in A$,

$$\text{if } x \in X, \text{ then } 0 \neq 0.$$

Suppose $x \in A$, and assume the inductive hypothesis: for all $y \in A$, if $y \mathrel{R} x$, then

$$\text{if } y \in X, \text{ then } 0 \neq 0.$$

We must show that

$$\text{if } x \in X, \text{ then } 0 \neq 0.$$

Suppose $x \in X$. We must show that $0 \neq 0$. Because $x \in X$, (†) tells us that there is a $y \in X$ such that $y \mathrel{R} x$. Thus the inductive hypothesis tells us that, if $y \in X$, then $0 \neq 0$. But $y \in X$, and so we can conclude $0 \neq 0$.

Now, we use the result of our well-founded induction to show that $0 \neq 0$. Because X is nonempty, there is an element x of X. By the result of our induction, we have that, if $x \in X$, then $0 \neq 0$. But $x \in X$, and thus $0 \neq 0$.