COMP 150-EFL—Experimenting with Formal Languages—Spring 2012

Problem Set 1

Model Answers

Problem 1

Define a function $f \in \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by: for all $n \in \mathbb{N}$, $f\,n = (n, 0)$. To see that f is injective, suppose $n, m \in \mathbb{N}$ and $f\,n = f\,m$. Then $(n, 0) = f\,n = f\,m = (m, 0)$, so that $n = m$, as required. Thus $\mathbb{N} \preceq \mathbb{N} \times \mathbb{N}$. By the Schröder-Bernstein Theorem, it remains to show that $\mathbb{N} \times \mathbb{N} \preceq \mathbb{N}$.

Let $X = \{\, n \in \mathbb{N} \mid n \text{ is prime} \,\}$ and $Y = \{\, ns \in \mathbf{List}\, X \mid \text{for all } 1 \leq i < j \leq |ns|, ns\,i \leq ns\,j \,\}$. Define a function $g \in Y \to \mathbb{N}$ by: for all $ns \in Y$, $g\,ns$ is the product of ns. To see that g is injective, suppose $ns, ms \in Y$ and $g\,ns = g\,ms$. By the Fundamental Theorem of Arithmetic, we have that $ns = ms$. Thus g is injective, so that $Y \preceq \mathbb{N}$.

We define a function $\mathbf{rep} \in \mathbb{N} \times \mathbb{N} \to \mathbf{List}\,\mathbb{N}$ by recursion on its second argument:

$$\mathbf{rep}(n, 0) = [\,], \text{ for all } n \in \mathbb{N}, \text{ and}$$
$$\mathbf{rep}(n, m + 1) = [n] \mathbin{@} \mathbf{rep}(n, m), \text{ for all } n, m \in \mathbb{N}.$$

Define a function $h \in \mathbb{N} \times \mathbb{N} \to Y$ by: for all $n, m \in \mathbb{N}$, $h(n, m) = \mathbf{rep}(2, n) \mathbin{@} \mathbf{rep}(3, m)$. Because $2, 3 \in X$ and $2 \leq 3$, it follows that h is well-defined. To see that h is injective, suppose $n, n', m, m' \in \mathbb{N}$ and $h(n, m) = h(n', m')$. Thus $\mathbf{rep}(2, n) @ \mathbf{rep}(3, m) = h(n, m) = h(n', m') = \mathbf{rep}(2, n') @ \mathbf{rep}(3, m')$. Since $\mathbf{rep}(2, n)$ and $\mathbf{rep}(2, n')$ consist entirely of 2's, whereas $\mathbf{rep}(3, m)$ and $\mathbf{rep}(3, m')$ have no 2's, it follows that $n = n'$, so that $m = m'$. Thus h is injective, so that $\mathbb{N} \times \mathbb{N} \preceq Y$.

Finally, because $\mathbb{N} \times \mathbb{N} \preceq Y \preceq \mathbb{N}$, we have that $\mathbb{N} \times \mathbb{N} \preceq \mathbb{N}$, completing the proof.

Exercise 2

We proceed by strong induction. Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: for all $m \in \mathbb{N}$, if $m < n$, then,

$$\text{if } m \geq 18, \text{ then there are } i, j \in \mathbb{N} \text{ such that } m = 4i + 7j.$$

We must show that,

$$\text{if } n \geq 18, \text{ then there are } i, j \in \mathbb{N} \text{ such that } n = 4i + 7j.$$

Suppose $n \geq 18$. We must show that there are $i, j \in \mathbb{N}$ such that $n = 4i + 7j$. There are five cases to consider.

- Suppose $n = 18$. Then $n = 18 = 4 * 1 + 7 * 2$ and $1, 2 \in \mathbb{N}$.
- Suppose $n = 19$. Then $n = 19 = 4 * 3 + 7 * 1$ and $3, 1 \in \mathbb{N}$.
- Suppose $n = 20$. Then $n = 20 = 4 * 5 + 7 * 0$ and $5, 0 \in \mathbb{N}$.
- Suppose $n = 21$. Then $n = 21 = 4 * 0 + 7 * 3$ and $0, 3 \in \mathbb{N}$.

- Suppose $n \geq 22$. Thus $18 \leq n - 4 < n$. Because $n - 4 < n$, the inductive hypothesis tells us that

  $$\text{if } n - 4 \geq 18, \text{ then there are } i, j \in \mathbb{N} \text{ such that } n - 4 = 4i + 7j.$$

  But $n - 4 \geq 18$, and thus $n - 4 = 4i + 7j$ for some $i, j \in \mathbb{N}$. Hence

  $$n = (n - 4) + 4 = 4i + 7j + 4 = 4(i + 1) + 7j,$$

  and $i + 1, j \in \mathbb{N}$.

Exercise 3

For $n \in \mathbb{N}$, define $L_n = \{\, xs \in \textbf{List}\, A \mid |xs| = n \,\}$. And, for $n \in \mathbb{N}$, define a relation T_n on L_n by: T_n is the set of all (xs, ys) such that $xs, ys \in L_n$ and, there is an $i \in [1:n]$ such that

- for all $j \in [1 : i - 1]$, $xs\, j = ys\, j$, and
- $xs\, i\ R\ ys\, i$.

Lemma PS1.3.1

(1) $T_0 = \emptyset$.

(2) For all $n \in \mathbb{N}$, $x, y \in A$ *and* $xs, ys \in L_n$, $([x] \mathbin{@} xs)\ T_{n+1}\ ([y] \mathbin{@} ys)$ *iff either:*

- $x\ R\ y$*; or*
- $x = y$ *and* $xs\ T_n\ ys$.

Proof.

(1) Suppose, toward a contradiction, that $T_0 \neq \emptyset$. Since $L_0 = \{[\,]\}$, this means that $[\,]\ T_0\ [\,]$. Hence, by the definition of T_0, $[1 : 0]$ is nonempty—contradiction. This $T_0 = \emptyset$.

(2) Suppose $n \in \mathbb{N}$, $x, y \in A$ and $xs, ys \in L_n$. There are two directions to show.

(only if) Suppose $([x] \mathbin{@} xs)\ T_{n+1}\ ([y] \mathbin{@} ys)$. Then there is an $i \in [1 : n + 1]$ such that, for all $j \in [1 : i - 1]$, $([x] \mathbin{@} xs)\, j = ([y] \mathbin{@} ys)\, j$, and $([x] \mathbin{@} xs)\, i\ R\ ([y] \mathbin{@} ys)\, i$. If $i = 1$, then $x\ R\ y$, and we are done. So, suppose $i \geq 2$ (so that $n \geq 1$). Then $x = y$, $i - 1 \in [1 : n]$, for all $j \in [1 : (i - 1) - 1]$, $xs\, j = ys\, j$, and $xs\, (i - 1)\ R\ ys\, (i - 1)$. Hence $xs\ T_n\ ys$.

(if) If $x\ R\ y$, then $1 \in [1 : n + 1]$, for all $j \in [1 : 1 - 1]$, $([x] \mathbin{@} xs)\, j = ([y] \mathbin{@} ys)\, j$, and $([x] \mathbin{@} xs)\, 1\ R\ ([y] \mathbin{@} ys)\, 1$, showing that $([x] \mathbin{@} xs)\ T_{n+1}\ ([y] \mathbin{@} ys)$. So, suppose $x = y$ and $xs\ T_n\ ys$. Thus, there is an $i \in [1 : n]$ such that, for all $j \in [1 : i - 1]$, $xs\, j = ys\, j$, and $xs\, i\ R\ ys\, i$. Hence $i + 1 \in [1 : n + 1]$, for all $j \in [1 : (i + 1) - 1]$, $([x] \mathbin{@} xs)\, j = ([y] \mathbin{@} ys)\, j$, and $([x] \mathbin{@} xs)(i + 1)\ R\ ([y] \mathbin{@} ys)(i + 1)$, showing that $([x] \mathbin{@} xs)\ T_{n+1}\ ([y] \mathbin{@} ys)$.

$\square$

Lemma PS1.3.2

For all $n \in \mathbb{N}$, T_n *is a well-founded relation on* L_n.

Proof. We proceed by mathematical induction.

(Basis Step) We must show that T_0 is a well-founded relation on L_0. Let X be a nonempty subset of $L_0 = \{[\,]\}$ so that $X = \{[\,]\}$. It will suffice to show that $[\,]$ is T_0-minimal in X. Suppose, toward a contradiction, that $[\,]$ is not T_0-minimal in X. Then $[\,]\ T_0\ [\,]$. But $T_0 = \emptyset$, by Lemma PS1.3.1(1), so we have our contradiction. Thus $[\,]$ is T_0-minimal in X.

(Inductive Step) Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: T_n is a well-founded relation on L_n. We must show that T_{n+1} is a well-founded relation on L_{n+1}. Define a function $f \in L_{n+1} \to A \times L_n$ by:

$$f([x] \mathbin{@} xs) = (x, xs), \text{ for all } x \in A \text{ and } xs \in L_n.$$

Let $U = f^{-1}(R \triangleright T_n)$. Since R and T_n are well-founded relations on A and L_n, respectively, Proposition 1.2.11 tells us that $R \triangleright T_n$ is a well-founded relation on $A \times L_n$. Thus, by Proposition 1.2.10, U is a well-founded relation on L_{n+1}. By the definition of U, we have that, for all $x, y \in A$ and $xs, ys \in L_n$, $([x] \mathbin{@} xs)\ U\ ([y] \mathbin{@} ys)$ iff $f([x] \mathbin{@} xs)\ R \triangleright T_n\ f([y] \mathbin{@} ys)$ iff $(x, xs)\ R \triangleright T_n\ (y, ys)$ iff either

- $x\ R\ y$, or
- $x = y$ and $xs\ T_n\ ys$

iff $([x] \mathbin{@} xs)\ T_{n+1}\ ([y] \mathbin{@} ys)$, by Lemma PS1.3.1(2). Hence $U = T_{n+1}$, so that T_{n+1} is a well-founded relation on L_{n+1}.

□

Let

$$T = \bigcup \{\, T_n \mid n \in \mathbb{N} \,\}.$$

Thus T is a relation on **List** A, and, for all $xs, ys \in$ **List** A, $xs\ T\ ys$ iff $xs\ T_n\ ys$ for some $n \in \mathbb{N}$ iff $|xs| = |ys|$ and, there is an $i \in [1 : |xs|]$ such that

- for all $j \in [1 : i - 1]$, $xs\, j = ys\, j$, and
- $xs\, i\ R\ ys\, i$.

Lemma PS1.3.3
T is a well-founded relation on **List** *A.*

Proof. Suppose X is a nonempty subset of **List** A. We must show that X has a T-minimal element. Since $X \neq \emptyset$, there is an $xs \in X$. Let $n = |xs|$ and $Y = \{\, ys \in X \mid |ys| = n \,\}$. Because Y is a nonempty subset of L_n, and T_n is a well-founded relation on L_n (Lemma PS1.3.2), there is a $ys \in Y$ such that ys is T_n-minimal in Y. Suppose, toward a contradiction, that ys is not T-minimal in X. Then there is a $zs \in X$ such that $zs\ T\ ys$, so that $zs\ T_m\ ys$ for some $m \in \mathbb{N}$. But then $m = |zs| = |ys| = n$, so that $zs \in Y$ and $zs\ T_n\ ys$—contradiction. Thus ys is T-minimal in X, completing the proof that T is a well-founded relation on **List** A. □

Define a function $f \in$ **List** $A \to \mathbb{N} \times$ **List** A by: for all $xs \in$ **List** A, $f\ xs = (|xs|, xs)$. Let

$$U = f^{-1}(< \triangleright T),$$

were $<$ is the usual strict total ordering on $\mathbb{N}$. Because $<$ and T are well-founded relations on $\mathbb{N}$ and $\mathbf{List}\, A$, respectively, Proposition 1.2.11 tells us that $< \triangleright T$ is a well-founded relation on $\mathbb{N} \times \mathbf{List}\, A$. Thus U is a well-founded relation on $\mathbf{List}\, A$, by Proposition 1.2.10. By the definition of U, we have that, for all $xs, ys \in \mathbf{List}\, A$, $xs\ U\ ys$ iff $(f\ xs) < \triangleright T\ (f\ ys)$ iff $(|xs|, xs) < \triangleright T\ (|ys|, ys)$ iff:

- $|xs| < |ys|$; or
- $|xs| = |ys|$ and $xs\ T\ ys$.

Thus $S = U$, so that S is a well-founded relation on $\mathbf{List}\, A$, as required.

Exercise 4

Let X be a set, and suppose Y is a nonempty subset of $\mathbf{Tree}\, X$. We use the principle of induction on trees to show that, for all $tr \in \mathbf{Tree}\, X$, if $tr \in Y$, then Y has a $\mathbf{pred}_{\mathbf{Tree}\, X}$-minimal element. Suppose $x \in X$, $trs \in \mathbf{List}(\mathbf{Tree}\, X)$, and assume the inductive hypothesis: for all $i \in [1 : |trs|]$, if $trs\, i \in Y$, then Y has a $\mathbf{pred}_{\mathbf{Tree}\, X}$-minimal element. We must show that, if $(x, trs) \in Y$, then Y has a $\mathbf{pred}_{\mathbf{Tree}\, X}$-minimal element. Suppose $(x, trs) \in Y$. We must show that Y has a $\mathbf{pred}_{\mathbf{Tree}\, X}$-minimal element. If (x, trs) is $\mathbf{pred}_{\mathbf{Tree}\, X}$-minimal in Y, then we are done. Otherwise, there is a $tr \in Y$ such that $tr\ \mathbf{pred}_{\mathbf{Tree}\, X}\ (x, trs)$. By the definition of $\mathbf{pred}_{\mathbf{Tree}\, X}$, it follows that $tr = trs\, i$ for some $i \in [1 : |trs|]$. Then, since $trs\, i \in Y$, the inductive hypothesis tells us that Y has a $\mathbf{pred}_{\mathbf{Tree}\, X}$-minimal element, as required.