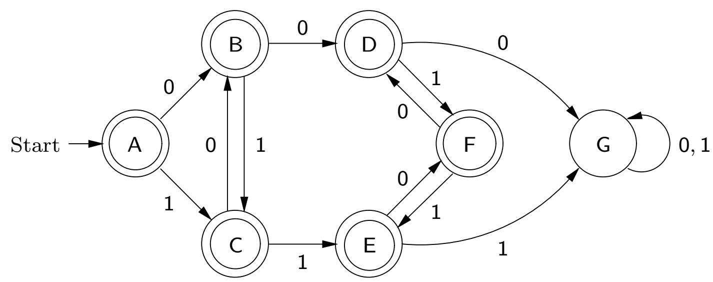
COMP 150-EFL—Experimenting with Formal Languages—Spring 2012

Problem Set 4

Model Answers

Problem 1

M is



Problem 2

First, we put the description of M in the file `ps4-dfa`:

```
{states} A, B, C, D, E, F, G
{start state} A
{accepting states} A, B, C, D, E, F
{transitions}
A, 0 -> B; A, 1 -> C;
B, 0 -> D; B, 1 -> C;
C, 0 -> B; C, 1 -> E;
D, 0 -> G; D, 1 -> F;
E, 0 -> F; E, 1 -> G;
F, 0 -> D; F, 1 -> E;
G, 0 -> G; G, 1 -> G
```

We then input M into Forlan, binding it to `dfa`:

```
- val dfa = DFA.input "ps4-dfa";
val dfa = - : dfa
```

Next, we put a testing harness in the file `ps4-prob2.sml`:

```
(* val zero : sym
   val one  : sym

   the symbols 0 and 1 *)

val zero = Sym.fromString "0";
val one  = Sym.fromString "1";
```

```
(* val diff : str -> int

   the diff function *)

fun diff (nil : str) = 0
  | diff (b :: bs)   =
      if Sym.equal(b, zero)
      then ~1 + diff bs
      else 1 + diff bs;

(* val prefixes : str -> str set

   returns the set of all prefixes of a string *)

fun prefixes (nil : str) : str set = Set.sing nil
  | prefixes (b :: bs)             =
      StrSet.union
      (Set.sing nil,
       StrSet.map (fn cs => b :: cs) (prefixes bs));

(* val substrings : str -> str set

   returns the set of all substrings of a string *)

fun substrings (nil : str) : str set = Set.sing nil
  | substrings (b :: bs)             =
      StrSet.union
      (substrings bs,
       StrSet.map (fn cs => b :: cs) (prefixes bs));

(* val allSubstringsGood : str -> bool

   tests whether all substrings of a string have diff's between
   ~2 and 2, inclusive *)

fun allSubstringsGood x =
      Set.all
      (fn y =>
            let val n = diff y
            in ~2 <= n andalso n <= 2 end)
      (substrings x);

(* val upto : int -> str set

   if n >= 0, then upto n returns all strings over the alphabet {0, 1}
   of length no more than n *)
```

```
fun upto 0 : str set = Set.sing nil
  | upto n           =
      let val xs = upto(n - 1)
          val ys = Set.filter (fn x => length x = n - 1) xs
      in StrSet.union
         (xs, StrSet.concat(StrSet.fromString "0, 1", ys))
      end;

(* val partition : int -> str set * str set

   if n >= 0, then partition n returns (xs, ys) where:

   xs is all elements w of upto n such that every substring of w
   has a diff between ~2 and 2, inclusive; and

   ys is all elements w of upto n such that at least one substring of
   w has a diff that's not between ~2 and 2, inclusive *)

fun partition n = Set.partition allSubstringsGood (upto n);

(* val test : int -> int * int * (dfa -> bool)

   if n >= 0, then test n returns

     (m, l, f) : int * int * (dfa -> bool),

   where m is the size of #1(partition n), l is the size of
   #2(partition n), and f is the function that takes in a DFA dfa, and
   tests whether dfa accepts all the elements of #1(partition n), but
   rejects all the elements of #2(partition n) *)

fun test n =
      let val (goods, bads) = partition n
      in (Set.size goods, Set.size bads,
          fn dfa =>
               let val determAccepted = DFA.determAccepted dfa
               in Set.all determAccepted goods        andalso
                  Set.all (not o determAccepted) bads
               end)
      end;
```

We then load the testing harness into Forlan:

```
- use "ps4-prob2.sml";
[opening ps4-prob2.sml]
val zero = - : sym
val one = - : sym
val diff = fn : str -> int
val prefixes = fn : str -> str set
```

```
val substrings = fn : str -> str set
val allSubstringsGood = fn : str -> bool
val upto = fn : int -> str set
val partition = fn : int -> str set * str set
val test = fn : int -> int * int * (dfa -> bool)
val it = () : unit
```

Finally, we test M on all 65535 ($2^{16} - 1$) strings of length up to 15 as follows:

```
- val (l, m, doit) = test 15;
val l = 1753 : int
val m = 63782 : int
val doit = fn : dfa -> bool
- doit dfa;
val it = true : bool
```

Problem 3

Define a function **dsfxs** (for "diffs of suffixes") from $\{0, 1\}^*$ to $\mathcal{P}\,\mathbb{Z}$ by: for all $w \in \{0, 1\}^*$,

$$\mathbf{dsfxs}\, w = \{\, \mathbf{diff}\, v \mid v \text{ is a suffix of } w \,\}.$$

Given $X \subseteq \mathbb{Z}$, we write

$$X^+ = \{\, n + 1 \mid n \in X \,\}, \text{ and}$$
$$X^- = \{\, n - 1 \mid n \in X \,\}.$$

Thus, we have that, for all $X \subseteq \mathbb{Z}$:

- $n \in X$ iff $n + 1 \in X^+$, for all $n \in \mathbb{Z}$;
- $n - 1 \in X$ iff $n \in X^+$, for all $n \in \mathbb{Z}$;
- $n \in X$ iff $n - 1 \in X^-$, for all $n \in \mathbb{Z}$; and
- $n + 1 \in X$ iff $n \in X^-$, for all $n \in \mathbb{Z}$.

Lemma PS4.3.1

(1) $\% \in X$.

(2) For all $w \in \{0, 1\}^*$, *if* $w \in X$ *and* $-2 \notin \mathbf{dsfxs}\, w$, *then* $w0 \in X$.

(3) For all $w \in \{0, 1\}^*$, *if* $w \in X$ *and* $2 \notin \mathbf{dsfxs}\, w$, *then* $w1 \in X$.

Proof.

(1) Follows since $\%$ is the only substring of $\%$, $\mathbf{diff}\, \% = 0$ and $-2 \leq 0 \leq 2$.

(2) Suppose $w \in \{0, 1\}^*$, $w \in X$ and $-2 \notin \mathbf{dsfxs}\, w$. To see that $w0 \in X$, suppose v is a substring of $w0$. We must show that $-2 \leq \mathbf{diff}\, v \leq 2$. There are two cases to consider (if $v = \%$, then v is a substring of w).

- Suppose v is a substring of w. Then $-2 \le \textbf{diff}\, v \le 2$, since $w \in X$.
- Suppose $v = u\mathsf{0}$, for a suffix u of w. Thus $\textbf{diff}\, v = \textbf{diff}(u\mathsf{0}) = \textbf{diff}\, u + \textbf{diff}\, \mathsf{0} = \textbf{diff}\, u + -1$. Because $w \in X$ and u is a substring of w, we have that $-2 \le \textbf{diff}\, u \le 2$. But $-2 \notin \textbf{dsfxs}\, w$, and so $\textbf{diff}\, u \ne -2$. Hence $-1 \le \textbf{diff}\, u \le 2$. Thus $-1 + -1 \le \textbf{diff}\, u + -1 \le 2 + -1$, so that $-2 = -1 + -1 \le \textbf{diff}\, v \le 2 + -1 = 1 \le 2$.

(3) Suppose $w \in \{\mathsf{0}, \mathsf{1}\}^*$, $w \in X$ and $2 \notin \textbf{dsfxs}\, w$. To see that $w\mathsf{1} \in X$, suppose v is a substring of $w\mathsf{1}$. We must show that $-2 \le \textbf{diff}\, v \le 2$. There are two cases to consider (if $v = \%$, then v is a substring of w).

- Suppose v is a substring of w. Then $-2 \le \textbf{diff}\, v \le 2$, since $w \in X$.
- Suppose $v = u\mathsf{1}$, for a suffix u of w. Thus $\textbf{diff}\, v = \textbf{diff}(u\mathsf{1}) = \textbf{diff}\, u + \textbf{diff}\, \mathsf{1} = \textbf{diff}\, u + 1$. Because $w \in X$ and u is a substring of w, we have that $-2 \le \textbf{diff}\, u \le 2$. But $2 \notin \textbf{dsfxs}\, w$, and so $\textbf{diff}\, u \ne 2$. Hence $-2 \le \textbf{diff}\, u \le 1$. Thus $-2 + 1 \le \textbf{diff}\, u + 1 \le 1 + 1$, so that $-2 \le -1 = -2 + 1 \le \textbf{diff}\, v \le 1 + 1 = 2$.

□

Lemma PS4.3.2

(1) $\textbf{dsfxs}\, \% = \{0\}$

(2) For all $w \in \{\mathsf{0}, \mathsf{1}\}^*$, $\textbf{dsfxs}(w\mathsf{0}) = (\textbf{dsfxs}\, w)^- \cup \{0\}$.

(3) For all $w \in \{\mathsf{0}, \mathsf{1}\}^*$, $\textbf{dsfxs}(w\mathsf{1}) = (\textbf{dsfxs}\, w)^+ \cup \{0\}$.

Proof.

(1) We show that $\textbf{dsfxs}\, \% \subseteq \{0\} \subseteq \textbf{dsfxs}\, \%$.

($\textbf{dsfxs}\, \% \subseteq \{0\}$) Suppose $n \in \textbf{dsfxs}\, \%$. Then $n = \textbf{diff}\, v$ for some suffix v of $\%$. Hence $v = \%$, so that $n = \textbf{diff}\, v = \textbf{diff}\, \% = 0 \in \{0\}$.

($\{0\} \subseteq \textbf{dsfxs}\, \%$) Since $\%$ is a suffix of $\%$, we have that $0 = \textbf{diff}\, \% \in \textbf{dsfxs}\, \%$. Hence $\{0\} \subseteq \textbf{dsfxs}\, \%$.

(2) Suppose $w \in \{\mathsf{0}, \mathsf{1}\}^*$. We show that $\textbf{dsfxs}(w\mathsf{0}) \subseteq (\textbf{dsfxs}\, w)^- \cup \{0\} \subseteq \textbf{dsfxs}(w\mathsf{0})$.

($\textbf{dsfxs}(w\mathsf{0}) \subseteq (\textbf{dsfxs}\, w)^- \cup \{0\}$) Suppose $n \in \textbf{dsfxs}(w\mathsf{0})$. Hence $n = \textbf{diff}\, v$ for some suffix v of $w\mathsf{0}$. There are two cases to consider.

- Suppose $v = \%$. Then $n = \textbf{diff}\, v = \textbf{diff}\, \% = 0 \in \{0\} \subseteq (\textbf{dsfxs}\, w)^- \cup \{0\}$.
- Suppose $v = u\mathsf{0}$, for some suffix u of w. Thus $n = \textbf{diff}\, v = \textbf{diff}(u\mathsf{0}) = \textbf{diff}\, u + \textbf{diff}\, \mathsf{0} = \textbf{diff}\, u + -1 = \textbf{diff}\, u - 1$. Since u is a suffix of w, we have that $\textbf{diff}\, u \in \textbf{dsfxs}\, w$, so that $\textbf{diff}\, u - 1 \in (\textbf{dsfxs}\, w)^-$. Hence $n = \textbf{diff}\, u - 1 \in (\textbf{dsfxs}\, w)^- \subseteq (\textbf{dsfxs}\, w)^- \cup \{0\}$.

($(\textbf{dsfxs}\, w)^- \cup \{0\} \subseteq \textbf{dsfxs}(w\mathsf{0})$) Suppose $n \in (\textbf{dsfxs}\, w)^- \cup \{0\}$. There are two cases to consider.

- Suppose $n \in (\textbf{dsfxs}\, w)^-$. Thus $n + 1 \in \textbf{dsfxs}\, w$, so that $n + 1 = \textbf{diff}\, v$, for some suffix v of w. Thus $v\mathsf{0}$ is a suffix of $w\mathsf{0}$, so that $n = \textbf{diff}\, v - 1 = \textbf{diff}\, v + -1 = \textbf{diff}\, v + \textbf{diff}\, \mathsf{0} = \textbf{diff}(v\mathsf{0}) \in \textbf{dsfxs}(w\mathsf{0})$.

- Suppose $n \in \{0\}$. Since $\%$ is a suffix of $w\mathsf{0}$, we have that $n = 0 = \mathbf{diff}\,\% \in \mathbf{dsfxs}(w\mathsf{0})$.

(3) Suppose $w \in \{\mathsf{0},\mathsf{1}\}^*$. We show that $\mathbf{dsfxs}(w\mathsf{1}) \subseteq (\mathbf{dsfxs}\,w)^+ \cup \{0\} \subseteq \mathbf{dsfxs}(w\mathsf{1})$.

($\mathbf{dsfxs}(w\mathsf{1}) \subseteq (\mathbf{dsfxs}\,w)^+ \cup \{0\}$) Suppose $n \in \mathbf{dsfxs}(w\mathsf{1})$. Hence $n = \mathbf{diff}\,v$ for some suffix v of $w\mathsf{1}$. There are two cases to consider.

- Suppose $v = \%$. Then $n = \mathbf{diff}\,v = \mathbf{diff}\,\% = 0 \in \{0\} \subseteq (\mathbf{dsfxs}\,w)^+ \cup \{0\}$.
- Suppose $v = u\mathsf{1}$, for some suffix u of w. Thus $n = \mathbf{diff}\,v = \mathbf{diff}(u\mathsf{1}) = \mathbf{diff}\,u + \mathbf{diff}\,\mathsf{1} = \mathbf{diff}\,u + 1$. Since u is a suffix of w, we have that $\mathbf{diff}\,u \in \mathbf{dsfxs}\,w$, so that $\mathbf{diff}\,u + 1 \in (\mathbf{dsfxs}\,w)^+$. Hence $n = \mathbf{diff}\,u + 1 \in (\mathbf{dsfxs}\,w)^+ \subseteq (\mathbf{dsfxs}\,w)^+ \cup \{0\}$.

($(\mathbf{dsfxs}\,w)^+ \cup \{0\} \subseteq \mathbf{dsfxs}(w\mathsf{1})$) Suppose $n \in (\mathbf{dsfxs}\,w)^+ \cup \{0\}$. There are two cases to consider.

- Suppose $n \in (\mathbf{dsfxs}\,w)^+$. Thus $n - 1 \in \mathbf{dsfxs}\,w$, so that $n - 1 = \mathbf{diff}\,v$, for some suffix v of w. Thus $v\mathsf{1}$ is a suffix of $w\mathsf{1}$, so that $n = \mathbf{diff}\,v + 1 = \mathbf{diff}\,v + \mathbf{diff}\,\mathsf{1} = \mathbf{diff}(v\mathsf{1}) \in \mathbf{dsfxs}(w\mathsf{1})$.
- Suppose $n \in \{0\}$. Since $\%$ is a suffix of $w\mathsf{1}$, we have that $n = 0 = \mathbf{diff}\,\% \in \mathbf{dsfxs}(w\mathsf{1})$.

$\square$

Lemma PS4.3.3

(A) For all $w \in \Lambda_{\mathsf{A}}$, $w \in X$ and $\mathbf{dsfxs}\,w = \{0\}$.

(B) For all $w \in \Lambda_{\mathsf{B}}$, $w \in X$ and $\mathbf{dsfxs}\,w = \{-1, 0\}$.

(C) For all $w \in \Lambda_{\mathsf{C}}$, $w \in X$ and $\mathbf{dsfxs}\,w = \{0, 1\}$.

(D) For all $w \in \Lambda_{\mathsf{D}}$, $w \in X$ and $\mathbf{dsfxs}\,w = \{-2, -1, 0\}$.

(E) For all $w \in \Lambda_{\mathsf{E}}$, $w \in X$ and $\mathbf{dsfxs}\,w = \{0, 1, 2\}$.

(F) For all $w \in \Lambda_{\mathsf{F}}$, $w \in X$ and $\mathbf{dsfxs}\,w = \{-1, 0, 1\}$.

(G) For all $w \in \Lambda_{\mathsf{G}}$, $w \notin X$.

Proof. We proceed by induction on Λ. There are 15 (1 plus the number of transitions) parts to show.

(empty string) We must show that $\% \in X$ and $\mathbf{dsfxs}\,\% = \{0\}$, which follow by Lemma PS4.3.1(1) and Lemma PS4.3.2(1).

($\mathsf{A}, \mathsf{0} \to \mathsf{B}$) Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\,w = \{0\}$. We must show that $w\mathsf{0} \in X$ and $\mathbf{dsfxs}(w\mathsf{0}) = \{-1, 0\}$. Since $w \in X$ and $-2 \notin \mathbf{dsfxs}\,w$, Lemma PS4.3.1(2) tells us that $w\mathsf{0} \in X$. And, by Lemma PS4.3.2(2), we have that $\mathbf{dsfxs}(w\mathsf{0}) = (\mathbf{dsfxs}\,w)^- \cup \{0\} = \{0\}^- \cup \{0\} = \{-1\} \cup \{0\} = \{-1, 0\}$.

($\mathsf{A}, \mathsf{1} \to \mathsf{C}$) Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\,w = \{0\}$. We must show that $w\mathsf{1} \in X$ and $\mathbf{dsfxs}(w\mathsf{1}) = \{0, 1\}$. Since $w \in X$ and $2 \notin \mathbf{dsfxs}\,w$, Lemma PS4.3.1(3) tells us that $w\mathsf{1} \in X$. And, by Lemma PS4.3.2(3), we have that $\mathbf{dsfxs}(w\mathsf{1}) = (\mathbf{dsfxs}\,w)^+ \cup \{0\} = \{0\}^+ \cup \{0\} = \{1\} \cup \{0\} = \{0, 1\}$.

$(\mathsf{B}, 0 \to \mathsf{D})$ Suppose $w \in \Lambda_{\mathsf{B}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{-1, 0\}$. We must show that $w\mathsf{0} \in X$ and $\mathbf{dsfxs}(w\mathsf{0}) = \{-2, -1, 0\}$. Since $w \in X$ and $-2 \notin \mathbf{dsfxs}\, w$, Lemma PS4.3.1(2) tells us that $w\mathsf{0} \in X$. And, by Lemma PS4.3.2(2), we have that $\mathbf{dsfxs}(w\mathsf{0}) = (\mathbf{dsfxs}\, w)^{-} \cup \{0\} = \{-1, 0\}^{-} \cup \{0\} = \{-2, -1\} \cup \{0\} = \{-2, -1, 0\}$.

$(\mathsf{B}, 1 \to \mathsf{C})$ Suppose $w \in \Lambda_{\mathsf{B}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{-1, 0\}$. We must show that $w\mathsf{1} \in X$ and $\mathbf{dsfxs}(w\mathsf{1}) = \{0, 1\}$. Since $w \in X$ and $2 \notin \mathbf{dsfxs}\, w$, Lemma PS4.3.1(3) tells us that $w\mathsf{1} \in X$. And, by Lemma PS4.3.2(3), we have that $\mathbf{dsfxs}(w\mathsf{1}) = (\mathbf{dsfxs}\, w)^{+} \cup \{0\} = \{-1, 0\}^{+} \cup \{0\} = \{0, 1\} \cup \{0\} = \{0, 1\}$.

$(\mathsf{C}, 0 \to \mathsf{B})$ Suppose $w \in \Lambda_{\mathsf{C}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{0, 1\}$. We must show that $w\mathsf{0} \in X$ and $\mathbf{dsfxs}(w\mathsf{0}) = \{-1, 0\}$. Since $w \in X$ and $-2 \notin \mathbf{dsfxs}\, w$, Lemma PS4.3.1(2) tells us that $w\mathsf{0} \in X$. And, by Lemma PS4.3.2(2), we have that $\mathbf{dsfxs}(w\mathsf{0}) = (\mathbf{dsfxs}\, w)^{-} \cup \{0\} = \{0, 1\}^{-} \cup \{0\} = \{-1, 0\} \cup \{0\} = \{-1, 0\}$.

$(\mathsf{C}, 1 \to \mathsf{E})$ Suppose $w \in \Lambda_{\mathsf{C}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{0, 1\}$. We must show that $w\mathsf{1} \in X$ and $\mathbf{dsfxs}(w\mathsf{1}) = \{0, 1, 2\}$. Since $w \in X$ and $2 \notin \mathbf{dsfxs}\, w$, Lemma PS4.3.1(3) tells us that $w\mathsf{1} \in X$. And, by Lemma PS4.3.2(3), we have that $\mathbf{dsfxs}(w\mathsf{1}) = (\mathbf{dsfxs}\, w)^{+} \cup \{0\} = \{0, 1\}^{+} \cup \{0\} = \{1, 2\} \cup \{0\} = \{0, 1, 2\}$.

$(\mathsf{D}, 0 \to \mathsf{G})$ Suppose $w \in \Lambda_{\mathsf{D}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{-2, -1, 0\}$. We must show that $w\mathsf{0} \notin X$. Since $-2 \in \mathbf{dsfxs}\, w$, there is a suffix v of w such that $\mathbf{diff}\, v = -2$. Thus $v\mathsf{0}$ is a suffix of $w\mathsf{0}$, and $\mathbf{diff}(v\mathsf{0}) = \mathbf{diff}\, v + \mathbf{diff}\, \mathsf{0} = -2 + -1 = -3$. Since $v\mathsf{0}$ is a substring of $w\mathsf{0}$ and $\mathbf{diff}(v\mathsf{0}) = -3$, it follows that $w\mathsf{0} \notin X$.

$(\mathsf{D}, 1 \to \mathsf{F})$ Suppose $w \in \Lambda_{\mathsf{D}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{-2, -1, 0\}$. We must show that $w\mathsf{1} \in X$ and $\mathbf{dsfxs}(w\mathsf{1}) = \{-1, 0, 1\}$. Since $w \in X$ and $2 \notin \mathbf{dsfxs}\, w$, Lemma PS4.3.1(3) tells us that $w\mathsf{1} \in X$. And, by Lemma PS4.3.2(3), we have that $\mathbf{dsfxs}(w\mathsf{1}) = (\mathbf{dsfxs}\, w)^{+} \cup \{0\} = \{-2, -1, 0\}^{+} \cup \{0\} = \{-1, 0, 1\} \cup \{0\} = \{-1, 0, 1\}$.

$(\mathsf{E}, 0 \to \mathsf{F})$ Suppose $w \in \Lambda_{\mathsf{E}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{0, 1, 2\}$. We must show that $w\mathsf{0} \in X$ and $\mathbf{dsfxs}(w\mathsf{0}) = \{-1, 0, 1\}$. Since $w \in X$ and $-2 \notin \mathbf{dsfxs}\, w$, Lemma PS4.3.1(2) tells us that $w\mathsf{0} \in X$. And, by Lemma PS4.3.2(2), we have that $\mathbf{dsfxs}(w\mathsf{0}) = (\mathbf{dsfxs}\, w)^{-} \cup \{0\} = \{0, 1, 2\}^{-} \cup \{0\} = \{-1, 0, 1\} \cup \{0\} = \{-1, 0, 1\}$.

$(\mathsf{E}, 1 \to \mathsf{G})$ Suppose $w \in \Lambda_{\mathsf{E}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{0, 1, 2\}$. We must show that $w\mathsf{1} \notin X$. Since $2 \in \mathbf{dsfxs}\, w$, there is a suffix v of w such that $\mathbf{diff}\, v = 2$. Thus $v\mathsf{1}$ is a suffix of $w\mathsf{1}$, and $\mathbf{diff}(v\mathsf{1}) = \mathbf{diff}\, v + \mathbf{diff}\, \mathsf{1} = 2 + 1 = 3$. Since $v\mathsf{1}$ is a substring of $w\mathsf{1}$ and $\mathbf{diff}(v\mathsf{1}) = 3$, it follows that $w\mathsf{1} \notin X$.

$(\mathsf{F}, 0 \to \mathsf{D})$ Suppose $w \in \Lambda_{\mathsf{F}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{-1, 0, 1\}$. We must show that $w\mathsf{0} \in X$ and $\mathbf{dsfxs}(w\mathsf{0}) = \{-2, -1, 0\}$. Since $w \in X$ and $-2 \notin \mathbf{dsfxs}\, w$, Lemma PS4.3.1(2) tells us that $w\mathsf{0} \in X$. And, by Lemma PS4.3.2(2), we have that $\mathbf{dsfxs}(w\mathsf{0}) = (\mathbf{dsfxs}\, w)^{-} \cup \{0\} = \{-1, 0, 1\}^{-} \cup \{0\} = \{-2, -1, 0\} \cup \{0\} = \{-2, -1, 0\}$.

$(\mathsf{F}, 1 \to \mathsf{E})$ Suppose $w \in \Lambda_{\mathsf{F}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{-1, 0, 1\}$. We must show that $w\mathsf{1} \in X$ and $\mathbf{dsfxs}(w\mathsf{1}) = \{0, 1, 2\}$. Since $w \in X$ and $2 \notin \mathbf{dsfxs}\, w$, Lemma PS4.3.1(3) tells us that $w\mathsf{1} \in X$. And, by Lemma PS4.3.2(3), we have that $\mathbf{dsfxs}(w\mathsf{1}) = (\mathbf{dsfxs}\, w)^{+} \cup \{0\} = \{-1, 0, 1\}^{+} \cup \{0\} = \{0, 1, 2\} \cup \{0\} = \{0, 1, 2\}$.

($\mathsf{G}, \mathsf{0} \to \mathsf{G}$) Suppose $w \in \Lambda_{\mathsf{G}}$, and assume the inductive hypothesis: $w \notin X$. We must show that $w\mathsf{0} \notin X$. Since $w \notin X$ but $w \in \Lambda_{\mathsf{G}} \subseteq (\textbf{alphabet}\, M)^* = \{\mathsf{0}, \mathsf{1}\}^*$, it follows that there is a substring v of w such $-2 \leq \textbf{diff}\, v \leq 2$ is false. But v is also a substring of $w\mathsf{0}$, and so $w\mathsf{0} \notin X$.

($\mathsf{G}, \mathsf{1} \to \mathsf{G}$) Suppose $w \in \Lambda_{\mathsf{G}}$, and assume the inductive hypothesis: $w \notin X$. We must show that $w\mathsf{1} \notin X$. Since $w \notin X$ but $w \in \Lambda_{\mathsf{G}} \subseteq (\textbf{alphabet}\, M)^* = \{\mathsf{0}, \mathsf{1}\}^*$, it follows that there is a substring v of w such $-2 \leq \textbf{diff}\, v \leq 2$ is false. But v is also a substring of $w\mathsf{1}$, and so $w\mathsf{1} \notin X$.

□

Proposition PS4.3.4
$L(M) = X$.

Proof. We show that $L(M) \subseteq X \subseteq L(M)$.

($L(M) \subseteq X$) Suppose $w \in L(M)$. Because $A_M = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}\}$, we have that $w \in L(M) = \Lambda_{\mathsf{A}} \cup \Lambda_{\mathsf{B}} \cup \Lambda_{\mathsf{C}} \cup \Lambda_{\mathsf{D}} \cup \Lambda_{\mathsf{E}} \cup \Lambda_{\mathsf{F}}$. Thus, by Lemma PS4.3.3(A–F), we have that $w \in X$.

($X \subseteq L(M)$) Suppose $w \in X$. Since $X \subseteq \{\mathsf{0}, \mathsf{1}\}^*$, we have that $w \in \{\mathsf{0}, \mathsf{1}\}^*$. Suppose, toward a contradiction, that $w \notin L(M)$. Because $w \notin L(M) = \Lambda_{\mathsf{A}} \cup \Lambda_{\mathsf{B}} \cup \Lambda_{\mathsf{C}} \cup \Lambda_{\mathsf{D}} \cup \Lambda_{\mathsf{E}} \cup \Lambda_{\mathsf{F}}$ and $w \in \{\mathsf{0}, \mathsf{1}\}^* = (\textbf{alphabet}\, M)^* = \Lambda_{\mathsf{A}} \cup \Lambda_{\mathsf{B}} \cup \Lambda_{\mathsf{C}} \cup \Lambda_{\mathsf{D}} \cup \Lambda_{\mathsf{E}} \cup \Lambda_{\mathsf{F}} \cup \Lambda_{\mathsf{G}}$, we must have that $w \in \Lambda_{\mathsf{G}}$. But then Lemma PS4.3.3(G) tells us that $w \notin X$—contradiction. Thus $w \in L(M)$.

□