## COMP 150-EFL—Experimenting with Formal Languages—Spring 2012

# Problem Set 6

## Model Answers

**Problem 1**

Let $\Sigma = \{\mathsf{0}, \mathsf{1}, \mathsf{2}, \mathsf{3}, \mathsf{4}\}$, and define a function $\textbf{diff} \in \Sigma^* \to \mathbb{Z}$ by: for all $w \in \Sigma^*$,

$$\textbf{diff}\, w = \text{the number of } \mathsf{3}\text{'s in } w - \text{the number of } \mathsf{4}\text{'s in } w.$$

Thus:

- $\textbf{diff}\, \% = 0$;
- $\textbf{diff}\, \mathsf{0} = \textbf{diff}\, \mathsf{1} = \textbf{diff}\, \mathsf{2} = 0$;
- $\textbf{diff}\, \mathsf{3} = 1$;
- $\textbf{diff}\, \mathsf{4} = -1$; and
- for all $x, y \in \Sigma^*$, $\textbf{diff}(xy) = \textbf{diff}\, x + \textbf{diff}\, y$.

Furthermore, easy mathematical inductions show that, for all $n \in \mathbb{N}$:

- $\textbf{diff}(\mathsf{3}^n) = n$; and
- $\textbf{diff}(\mathsf{4}^n) = -n$.

**Lemma PS6.1.1**
*For all $n \in \mathbb{N}$, $\mathsf{3}^n\mathsf{2}\mathsf{4}^n \in \Pi_{\mathsf{A}}$.*

**Proof.** We proceed by mathematical induction.

**(Basis Step)** Since $\mathsf{A} \to \mathsf{2} \in P_G$, we have that $\mathsf{3}^0\mathsf{2}\mathsf{4}^0 = \%\mathsf{2}\% = \mathsf{2} \in \Pi_{\mathsf{A}}$.

**(Inductive Step)** Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis: $\mathsf{3}^n\mathsf{2}\mathsf{4}^n \in \Pi_{\mathsf{A}}$. Since $\mathsf{A} \to \mathsf{3A4} \in P_G$, $\mathsf{3} \in \Pi_{\mathsf{3}}$, $\mathsf{3}^n\mathsf{2}\mathsf{4}^n \in \Pi_{\mathsf{A}}$ and $\mathsf{4} \in \Pi_{\mathsf{4}}$, it follows that $\mathsf{3}^{n+1}\mathsf{2}\mathsf{4}^{n+1} = \mathsf{3}(\mathsf{3}^n\mathsf{2}\mathsf{4}^n)\mathsf{4} \in \Pi_{\mathsf{A}}$.

$\square$

**Lemma PS6.1.2**
*For all $w \in \Pi_{\mathsf{A}}$, $\textbf{diff}\, w = 0$.*

**Proof.** We proceed by induction on $\Pi$.

($\mathsf{A} \to \mathsf{A0A}$) Suppose $x, y \in \Pi_{\mathsf{A}}$, and assume the inductive hypothesis: $\textbf{diff}\, x = 0$ and $\textbf{diff}\, y = 0$. Then $\textbf{diff}(x\mathsf{0}y) = \textbf{diff}\, x + \textbf{diff}\, \mathsf{0} + \textbf{diff}\, y = 0 + 0 + 0 = 0$.

($\mathsf{A} \to \mathsf{A1A}$) Similar to the preceding case.

($\mathsf{A} \to 2$) We have that $\mathbf{diff}\,2 = 0$.

($\mathsf{A} \to 3\mathsf{A}4$) Suppose $x \in \Pi_{\mathsf{A}}$, and assume the inductive hypothesis: $\mathbf{diff}\,x = 0$. Then $\mathbf{diff}(3x4) = \mathbf{diff}\,3 + \mathbf{diff}\,x + \mathbf{diff}\,4 = 1 + 0 + -1 = 0$.

$\square$

Now we use our lemmas to prove that $L(G)$ is not regular. Suppose, toward a contradiction, that $L(G)$ is regular. Thus there is an $n \in \mathbb{N}$ with the property of the Pumping Lemma, where $L(G)$ has been substituted for $L$. Suppose $z = 3^n 2 4^n$. By Lemma PS6.1.1, we have that $z \in L(G)$. Thus, since $|z| = 2n + 1 \geq n$, it follows that there are $u, v, w \in \mathbf{Str}$ such that $z = uvw$ and properties (1)–(3) of the lemma hold. Since $3^n 2 4^n = z = uvw$, (1) tells us that there are $i, j, k \in \mathbb{N}$ such that

$$u = 3^i, \quad v = 3^j, \quad w = 3^k 2 4^n, \quad i + j + k = n.$$

By (2), we have that $j \geq 1$, and thus that $i + k = n - j < n$. By (3), we have that

$$3^{i+k} 2 4^n = 3^i 3^k 2 4^n = uw = u\%w = uv^0 w \in L(G).$$

By Lemma PS6.1.2, we have that $i + k + -n = (i + k) + 0 + -n = \mathbf{diff}(3^{i+k}) + \mathbf{diff}\,2 + \mathbf{diff}(4^n) = \mathbf{diff}(3^{i+k} 2 4^n) = 0$. Thus $i + k = n$—contradiction. Thus $L(G)$ is not regular.

**Problem 2**

We put the description of $G$ in the file `ps6-gram1`:

```
{variables} A {start variable} A
{productions} A -> A0A | A1A | 2 | 3A4
```

We put the description of $H$ in the file `ps6-gram2`:

```
{variables} A, B, C {start variable} A
{productions}
A -> B | B0A;
B -> C | C1B;
C -> 2 | 3A4
```

And we load $G$ and $H$ into Forlan, binding $G$ to `gram1`, and $H$ to `gram2`, and then check that $\mathbf{alphabet}\,G = \mathbf{alphabet}\,H = \Sigma$:

```
- val gram1 = Gram.input "ps6-gram1";
val gram1 = - : gram
- val gram2 = Gram.input "ps6-gram2";
val gram2 = - : gram
- SymSet.output("", Gram.alphabet gram1);
0, 1, 2, 3, 4
val it = () : unit
- SymSet.output("", Gram.alphabet gram2);
0, 1, 2, 3, 4
val it = () : unit
```

Next, we put our testing harness in the file `ps6.sml`:

```
val regToDFA  = nfaToDFA o efaToNFA o faToEFA o regToFA;
val minAndRen =
      DFA.renameStatesCanonically o DFA.minimize o
      DFA.renameStatesCanonically;

(* val lenReg : int -> reg

   if n >= 0, then lenReg n returns a regular expression generating
   {0, 1, 2, 3, 4} to the power n *)

fun lenReg n = Reg.power(Reg.fromString "0 + 1 + 2 + 3 + 4", n);

(* val lenDFA : int -> dfa

   if n >= 0, then lenDFA n returns a minimized DFA generating
   {0, 1, 2, 3, 4} to the power n *)

val lenDFA = minAndRen o regToDFA o lenReg;

(* val test : gram * gram * int -> bool

   if n >= 0, then test(gram1, gram2, n) checks whether, for all
   w in {0, 1, 2, 3, 4}*, if |w| <= n, then w is generated by gram1
   iff w is generated by gram2 *)

fun test(gram1, gram2, n) =
      let fun loop m =
                m > n orelse
                let val efa  = injDFAToEFA(lenDFA m)
                    val gen1 = Gram.toStrSet(Gram.inter(gram1, efa))
                    val gen2 = Gram.toStrSet(Gram.inter(gram2, efa))
                in StrSet.equal(gen1, gen2) andalso loop(m + 1) end
      in loop 0 end;
```

And we load `ps6.sml` into Forlan:

```
- use "ps6.sml";
[opening ps6.sml]
val regToDFA = fn : reg -> dfa
val minAndRen = fn : dfa -> dfa
val lenReg = fn : int -> reg
val lenDFA = fn : int -> dfa
val test = fn : gram * gram * int -> bool
val it = () : unit
```

Finally, we use the function `test` to check that, for all $w \in \Sigma^*$, if $|w| \leq 20$, then $w$ is generated by `gram1` iff $w$ is generated by `gram2`:

```
- test(gram1, gram2, 20);
val it = true : bool
```

(This took about 8 minutes on my laptop. Because there are about 119 trillion elements of $\Sigma^*$ with length at most 20, it wouldn't have been viable to have tested our grammars on all these strings, one after the other.)

**Problem 3**

To see that $L(H) \subseteq L(G)$, it will suffice to prove the following lemma:

**Lemma PS6.3.1**

*(A) For all* $w \in \Pi_{H,\mathsf{A}}$, $w \in \Pi_{G,\mathsf{A}}$.

*(B) For all* $w \in \Pi_{H,\mathsf{B}}$, $w \in \Pi_{G,\mathsf{A}}$.

*(C) For all* $w \in \Pi_{H,\mathsf{C}}$, $w \in \Pi_{G,\mathsf{A}}$.

**Proof.** By induction on $\Pi_H$. There are six productions to consider.

($\mathsf{A} \to \mathsf{B}$) Suppose $x \in \Pi_{H,\mathsf{B}}$, and assume the inductive hypothesis: $x \in \Pi_{G,\mathsf{A}}$. Then $x \in \Pi_{G,\mathsf{A}}$.

($\mathsf{A} \to \mathsf{B0A}$) Suppose $x \in \Pi_{H,\mathsf{B}}$ and $y \in \Pi_{H,\mathsf{A}}$, and assume the inductive hypothesis: $x, y \in \Pi_{G,\mathsf{A}}$. Because $\mathsf{A} \to \mathsf{A0A} \in P_G$, it follows that $x\mathsf{0}y \in \Pi_{G,\mathsf{A}}$.

($\mathsf{B} \to \mathsf{C}$) Suppose $x \in \Pi_{H,\mathsf{C}}$, and assume the inductive hypothesis: $x \in \Pi_{G,\mathsf{A}}$. Then $x \in \Pi_{G,\mathsf{A}}$.

($\mathsf{B} \to \mathsf{C1B}$) Suppose $x \in \Pi_{H,\mathsf{C}}$ and $y \in \Pi_{H,\mathsf{B}}$, and assume the inductive hypothesis: $x, y \in \Pi_{G,\mathsf{A}}$. Because $\mathsf{A} \to \mathsf{A1A} \in P_G$, it follows that $x\mathsf{1}y \in \Pi_{G,\mathsf{A}}$.

($\mathsf{C} \to \mathsf{2}$) Because $\mathsf{A} \to \mathsf{2} \in P_G$, we have that $\mathsf{2} \in \Pi_{G,\mathsf{A}}$.

($\mathsf{C} \to \mathsf{3A4}$) Suppose $x \in \Pi_{H,\mathsf{A}}$, and assume the inductive hypothesis: $x \in \Pi_{G,\mathsf{A}}$. Because $\mathsf{A} \to \mathsf{3A4} \in P_G$, it follows that $\mathsf{3}x\mathsf{4} \in \Pi_{G,\mathsf{A}}$.

□

Thus it remains to show that $L(G) \subseteq L(H)$.

**Lemma PS6.3.2**

*For all* $x, y \in \Pi_{H,\mathsf{A}}$, $x\mathsf{0}y \in \Pi_{H,\mathsf{A}}$.

**Proof.** Suppose $y \in \Pi_{\mathsf{A}}$. To show that, for all $x \in \Pi_{\mathsf{A}}$, $x\mathsf{0}y \in \Pi_{\mathsf{A}}$, it will suffice to show that, for all $x \in \Sigma^*$, if $x \in \Pi_{\mathsf{A}}$, then $x\mathsf{0}y \in \Pi_{\mathsf{A}}$. We proceed by strong string induction. Suppose $x \in \Sigma^*$, and assume the inductive hypothesis: for all $x' \in \Sigma^*$, if $x'$ is a proper substring of $x$, then, if $x' \in \Pi_{\mathsf{A}}$, then $x'\mathsf{0}y \in \Pi_{\mathsf{A}}$. Suppose $x \in \Pi_{\mathsf{A}}$. We must show that $x\mathsf{0}y \in \Pi_{\mathsf{A}}$. Since $x \in \Pi_{\mathsf{A}}$, and $\mathsf{A} \to \mathsf{B}$ and $\mathsf{A} \to \mathsf{B0A}$ are all the productions of $H$ whose left-sides are $\mathsf{A}$, there are two cases to consider.

($\mathsf{A} \to \mathsf{B}$) Suppose $x \in \Pi_{\mathsf{B}}$. Because $\mathsf{A} \to \mathsf{B0A} \in P_H$, $x \in \Pi_{\mathsf{B}}$ and $y \in \Pi_{\mathsf{A}}$, we have that $x\mathsf{0}y \in \Pi_{\mathsf{A}}$.

($\mathsf{A} \to \mathsf{B0A}$) Suppose $x = u0v$ for some $u \in \Pi_\mathsf{B}$ and $v \in \Pi_\mathsf{A}$. Since $v$ is a proper substring of $x$, and $v \in \Pi_\mathsf{A}$, the inductive hypothesis tells us that $v0y \in \Pi_\mathsf{A}$. Since $\mathsf{A} \to \mathsf{B0A} \in P_H$, $u \in \Pi_\mathsf{B}$ and $v0y \in \Pi_\mathsf{A}$, we have that $x0y = (u0v)0y = u0(v0y) \in \Pi_\mathsf{A}$.

□

**Lemma PS6.3.3**
*For all* $x, y \in \Pi_{H,\mathsf{B}}$, $x1y \in \Pi_{H,\mathsf{B}}$.

**Proof.** Suppose $y \in \Pi_\mathsf{B}$. To show that, for all $x \in \Pi_\mathsf{B}$, $x1y \in \Pi_\mathsf{B}$, it will suffice to show that, for all $x \in \Sigma^*$, if $x \in \Pi_\mathsf{B}$, then $x1y \in \Pi_\mathsf{B}$. We proceed by strong string induction. Suppose $x \in \Sigma^*$, and assume the inductive hypothesis: for all $x' \in \Sigma^*$, if $x'$ is a proper substring of $x$, then, if $x' \in \Pi_\mathsf{B}$, then $x'1y \in \Pi_\mathsf{B}$. Suppose $x \in \Pi_\mathsf{B}$. We must show that $x1y \in \Pi_\mathsf{B}$. Since $x \in \Pi_\mathsf{B}$, there are two cases to consider.

($\mathsf{B} \to \mathsf{C}$) Suppose $x \in \Pi_\mathsf{C}$. Because $\mathsf{B} \to \mathsf{C1B} \in P_H$, $x \in \Pi_\mathsf{C}$ and $y \in \Pi_\mathsf{B}$, we have that $x1y \in \Pi_\mathsf{B}$.

($\mathsf{B} \to \mathsf{C1B}$) Suppose $x = u1v$ for some $u \in \Pi_\mathsf{C}$ and $v \in \Pi_\mathsf{B}$. Since $v$ is a proper substring of $x$, and $v \in \Pi_\mathsf{B}$, the inductive hypothesis tells us that $v1y \in \Pi_\mathsf{B}$. Since $\mathsf{B} \to \mathsf{C1B} \in P_H$, $u \in \Pi_\mathsf{C}$ and $v1y \in \Pi_\mathsf{B}$, we have that $x1y = (u1v)1y = u1(v1y) \in \Pi_\mathsf{B}$.

□

**Lemma PS6.3.4**
*For all* $x, y \in \Pi_{H,\mathsf{A}}$, $x1y \in \Pi_{H,\mathsf{A}}$.

**Proof.** Suppose $y \in \Pi_\mathsf{A}$. To show that, for all $x \in \Pi_\mathsf{A}$, $x1y \in \Pi_\mathsf{A}$, it will suffice to show that, for all $x \in \Sigma^*$, if $x \in \Pi_\mathsf{A}$, then $x1y \in \Pi_\mathsf{A}$. We proceed by strong string induction. Suppose $x \in \Sigma^*$, and assume the inductive hypothesis: for all $x' \in \Sigma^*$, if $x'$ is a proper substring of $x$, then, if $x' \in \Pi_\mathsf{A}$, then $x'1y \in \Pi_\mathsf{A}$. Suppose $x \in \Pi_\mathsf{A}$. We must show that $x1y \in \Pi_\mathsf{A}$. Since $x \in \Pi_\mathsf{A}$, there are two cases to consider.

($\mathsf{A} \to \mathsf{B}$) Suppose $x \in \Pi_\mathsf{B}$. Since $y \in \Pi_\mathsf{A}$, there are two subcases to consider.

- ($\mathsf{A} \to \mathsf{B}$) Suppose $y \in \Pi_\mathsf{B}$. Since $x, y \in \Pi_\mathsf{B}$, Lemma PS6.3.3 tells us that $x1y \in \Pi_\mathsf{B}$. Because $\mathsf{A} \to \mathsf{B} \in P_H$ and $x1y \in \Pi_\mathsf{B}$, we have that $x1y \in \Pi_\mathsf{A}$.
- ($\mathsf{A} \to \mathsf{B0A}$) Suppose $y = u0v$ for some $u \in \Pi_\mathsf{B}$ and $v \in \Pi_\mathsf{A}$. Since $x, u \in \Pi_\mathsf{B}$, Lemma PS6.3.3 tells us that $x1u \in \Pi_\mathsf{B}$. Because $\mathsf{A} \to \mathsf{B0A} \in P_H$, $x1u \in \Pi_\mathsf{B}$ and $v \in \Pi_\mathsf{A}$, we have that $x1y = x1(u0v) = (x1u)0v \in \Pi_\mathsf{A}$.

($\mathsf{A} \to \mathsf{B0A}$) Suppose $x = u0v$ for some $u \in \Pi_\mathsf{B}$ and $v \in \Pi_\mathsf{A}$. Since $v$ is a proper substring of $x$, and $v \in \Pi_\mathsf{A}$, the inductive hypothesis tells us that $v1y \in \Pi_\mathsf{A}$. Since $\mathsf{A} \to \mathsf{B0A} \in P_H$, $u \in \Pi_\mathsf{B}$ and $v1y \in \Pi_\mathsf{A}$, we have that $x1y = (u0v)1y = u0(v1y) \in \Pi_\mathsf{A}$.

□

To show that $L(G) \subseteq L(H)$, it will suffice to show the following lemma:

**Lemma PS6.3.5**
*For all* $w \in \Pi_{G,\mathsf{A}}$, $w \in \Pi_{H,\mathsf{A}}$.

**Proof.** By induction on $\Pi_G$. There are four productions to consider.

($\mathsf{A} \to \mathsf{A0A}$) Suppose $x, y \in \Pi_{G,\mathsf{A}}$, and assume the inductive hypothesis: $x, y \in \Pi_{H,\mathsf{A}}$. By Lemma PS6.3.2, we have that $x\mathsf{0}y \in \Pi_{H,\mathsf{A}}$.

($\mathsf{A} \to \mathsf{A1A}$) Suppose $x, y \in \Pi_{G,\mathsf{A}}$, and assume the inductive hypothesis: $x, y \in \Pi_{H,\mathsf{A}}$. By Lemma PS6.3.4, we have that $x\mathsf{1}y \in \Pi_{H,\mathsf{A}}$.

($\mathsf{A} \to \mathsf{2}$) Since $\mathsf{C} \to \mathsf{2} \in P_H$, we have that $\mathsf{2} \in \Pi_{H,\mathsf{C}}$. Then, since $\mathsf{B} \to \mathsf{C} \in P_H$, we have that $\mathsf{2} \in \Pi_{H,\mathsf{B}}$. Then, since $\mathsf{A} \to \mathsf{B} \in P_H$, we have that $\mathsf{2} \in \Pi_{H,\mathsf{A}}$.

($\mathsf{A} \to \mathsf{3A4}$) Suppose $x \in \Pi_{G,\mathsf{A}}$, and assume the inductive hypothesis: $x \in \Pi_{H,\mathsf{A}}$. Since $\mathsf{C} \to \mathsf{3A4} \in P_H$, we have that $\mathsf{3}x\mathsf{4} \in \Pi_{H,\mathsf{C}}$. Then, since $\mathsf{B} \to \mathsf{C} \in P_H$, we have that $\mathsf{3}x\mathsf{4} \in \Pi_{H,\mathsf{B}}$. Then, since $\mathsf{A} \to \mathsf{B} \in P_H$, we have that $\mathsf{3}x\mathsf{4} \in \Pi_{H,\mathsf{A}}$.

$\square$

**Problem 4**

Let $X = \{\, w \in \Sigma^* \mid \mathbf{diff}\, w = 0$ and, for all prefixes $v$ of $w$, $\mathbf{diff}\, v \geq 0 \,\}$. Since $\mathbf{diff}\, \mathsf{0} = \mathbf{diff}\, \mathsf{1} = \mathbf{diff}\, \mathsf{2} = 0$ and $\mathbf{diff}\, \% = 0$, we have that $\mathsf{0}, \mathsf{1}, \mathsf{2} \in X$.

We will use the following lemma without reference:

**Lemma PS6.4.1**
*For all $x, y \in X$, $xy \in X$.*

**Proof.** Suppose $x, y \in X$. Thus $\mathbf{diff}(xy) = \mathbf{diff}\, x + \mathbf{diff}\, y = 0 + 0 = 0$. To finish the proof that $xy \in X$, suppose $v$ is a prefix of $xy$. We must show that $\mathbf{diff}\, v \geq 0$. Since $v$ is a prefix of $xy$, there are two cases to consider.

- Suppose $v$ is a prefix of $x$. Since $x \in X$, it follows that $\mathbf{diff}\, v \geq 0$.
- Suppose $v = xu$ for a prefix $u$ of $y$. Since $y \in X$, it follows that $\mathbf{diff}\, u \geq 0$. Thus $\mathbf{diff}\, v = \mathbf{diff}(xu) = \mathbf{diff}\, x + \mathbf{diff}\, u = 0 + \mathbf{diff}\, u = \mathbf{diff}\, u \geq 0$.

$\square$

**Lemma PS6.4.2**

*(A) For all $w \in \Pi_{H,\mathsf{A}}$, $w \in X$.*

*(B) For all $w \in \Pi_{H,\mathsf{B}}$, $w \in X$.*

*(C) For all $w \in \Pi_{H,\mathsf{C}}$, $w \in X$.*

**Proof.** By induction on $\Pi_H$. There are six productions to consider.

($\mathsf{A} \to \mathsf{B}$) Suppose $x \in \Pi_{\mathsf{B}}$, and assume the inductive hypothesis: $x \in X$. Then $x \in X$.

($\mathsf{A} \to \mathsf{B0A}$) Suppose $x \in \Pi_{\mathsf{B}}$ and $y \in \Pi_{\mathsf{A}}$, and assume the inductive hypothesis: $x, y \in X$. Since $x, \mathsf{0}, y \in X$, it follows that $x\mathsf{0}y \in X$.

($\mathsf{B} \rightarrow \mathsf{C}$) Suppose $x \in \Pi_{\mathsf{C}}$, and assume the inductive hypothesis: $x \in X$. Thus $x \in X$.

($\mathsf{B} \rightarrow \mathsf{C1B}$) Suppose $x \in \Pi_{\mathsf{C}}$ and $y \in \Pi_{\mathsf{B}}$, and assume the inductive hypothesis: $x, y \in X$. Since $x, \mathsf{1}, y \in X$, it follows that $x\mathsf{1}y \in X$.

($\mathsf{C} \rightarrow \mathsf{2}$) We have that $\mathsf{2} \in X$.

($\mathsf{C} \rightarrow \mathsf{3A4}$) Suppose $x \in \Pi_{\mathsf{A}}$, and assume the inductive hypothesis: $x \in X$. We have that $\mathbf{diff}(\mathsf{3}x\mathsf{4}) = \mathbf{diff}\,\mathsf{3} + \mathbf{diff}\,x + \mathbf{diff}\,\mathsf{4} = 1 + 0 + -1 = 0$. To complete the proof that $\mathsf{3}x\mathsf{4} \in X$, suppose $v$ is a prefix of $\mathsf{3}x\mathsf{4}$. We must show that $\mathbf{diff}\,v \geq 0$. Since $v$ is a prefix of $\mathsf{3}x\mathsf{4}$, there are three cases to consider.

- Suppose $v = \%$. Then $\mathbf{diff}\,v = \mathbf{diff}\,\% = 0 \geq 0$.
- Suppose $v = \mathsf{3}u$ for a prefix $u$ of $x$. Since $x \in X$, we have that $\mathbf{diff}\,u \geq 0$. Thus $\mathbf{diff}\,v = \mathbf{diff}(\mathsf{3}u) = \mathbf{diff}\,\mathsf{3} + \mathbf{diff}\,u = 1 + \mathbf{diff}\,u \geq 1 + 0 \geq 0$.
- Suppose $v = \mathsf{3}x\mathsf{4}$. Then $\mathbf{diff}\,v = \mathbf{diff}(\mathsf{3}x\mathsf{4}) = 0 \geq 0$.

□

**Lemma PS6.4.3**
*For all $x, y \in \Sigma^*$, if $x\mathsf{0}y \in \Pi_{H,\mathsf{C}}$, then $\mathbf{diff}\,x \geq 1$.*

**Proof.** Suppose $x, y \in \Sigma^*$ and $x\mathsf{0}y \in \Pi_{\mathsf{C}}$. There are two cases to consider.

($\mathsf{C} \rightarrow \mathsf{2}$) Suppose $x\mathsf{0}y = \mathsf{2}$. This is contradictory, and thus we can conclude $\mathbf{diff}\,x \geq 1$.

($\mathsf{C} \rightarrow \mathsf{3A4}$) Suppose $x\mathsf{0}y = \mathsf{3}w\mathsf{4}$ for some $w \in \Pi_{\mathsf{A}}$. Then $x \neq \%$ and $y \neq \%$, so that there are $x', y' \in \Sigma^*$ such that $x = \mathsf{3}x'$ and $y = y'\mathsf{4}$. Since $\mathsf{3}x'\mathsf{0}y'\mathsf{4} = x\mathsf{0}y = \mathsf{3}w\mathsf{4}$, we have that $w = x'\mathsf{0}y'$. Since $w \in \Pi_{\mathsf{A}}$ and $x'$ is a prefix of $w$, Lemma PS6.4.2 tells us that $\mathbf{diff}\,x' \geq 0$. Thus $\mathbf{diff}\,x = \mathbf{diff}(\mathsf{3}x') = \mathbf{diff}\,\mathsf{3} + \mathbf{diff}\,x' = 1 + \mathbf{diff}\,x' \geq 1 + 0 = 1$, showing that $\mathbf{diff}\,x \geq 1$.

□

**Lemma PS6.4.4**
*For all $x, y \in \Sigma^*$, if $x\mathsf{1}y \in \Pi_{H,\mathsf{C}}$, then $\mathbf{diff}\,x \geq 1$.*

**Proof.** The proof is identical to the preceding proof, except that all occurrences of $\mathsf{0}$ are replaced by occurrences of $\mathsf{1}$. □

**Lemma PS6.4.5**
*For all $x, y \in \Sigma^*$, if $x\mathsf{0}y \in \Pi_{H,\mathsf{B}}$, then $\mathbf{diff}\,x \geq 1$.*

**Proof.** It will suffice to show that, for all $w \in \Sigma^*$, if $w \in \Pi_{\mathsf{B}}$, then, for all $x, y \in \Sigma^*$, if $w = x\mathsf{0}y$, then $\mathbf{diff}\,x \geq 1$. We proceed by strong string induction. Suppose $w \in \Sigma^*$, and assume the inductive hypothesis: for all $v \in \Sigma^*$, if $v$ is a proper substring of $w$, then, if $v \in \Pi_{\mathsf{B}}$, then, for all $x, y \in \Sigma^*$, if $v = x\mathsf{0}y$, then $\mathbf{diff}\,x \geq 1$. Suppose $w \in \Pi_{\mathsf{B}}$, $x, y \in \Sigma^*$ and $w = x\mathsf{0}y$. We must show that $\mathbf{diff}\,x \geq 1$. Because $x\mathsf{0}y = w \in \Pi_{\mathsf{B}}$, there are two cases to consider.

($\mathsf{B} \rightarrow \mathsf{C}$) Suppose $x\mathsf{0}y \in \Pi_{\mathsf{C}}$. By Lemma PS6.4.3, we have that $\mathbf{diff}\,x \geq 1$.

($\mathsf{B} \to \mathsf{C1B}$) Suppose $x\mathsf{0}y = u\mathsf{1}v$ for some $u \in \Pi_{\mathsf{C}}$ and $v \in \Pi_{\mathsf{B}}$. Since $w = x\mathsf{0}y = u\mathsf{1}v$, we have that $v$ is a proper substring of $w$. Since $x\mathsf{0}y = u\mathsf{1}v$, there are two subcases to consider.

- Suppose $x$ is a prefix of $u$. Then $x$ is a proper prefix of $u$, as otherwise we'd have $\mathsf{0} = \mathsf{1}$. So $u = x\mathsf{0}t$ for some $t \in \Sigma^*$. But $x\mathsf{0}t = u \in \Pi_{\mathsf{C}}$, and thus Lemma PS6.4.3 tells us that $\mathbf{diff}\, x \geq 1$.
- Suppose $u$ is a prefix of $x$. Thus $u$ is a proper prefix of $x$, so that $x = u\mathsf{1}t$ for some $t \in \Sigma^*$. Since $u\mathsf{1}t\mathsf{0}y = x\mathsf{0}y = u\mathsf{1}v$, we have that $t\mathsf{0}y = v$. Since $v$ is a proper substring of $w$, $v \in \Pi_{\mathsf{B}}$ and $v = t\mathsf{0}y$, the inductive hypothesis tells us that $\mathbf{diff}\, t \geq 1$. Because $u \in \Pi_{\mathsf{C}}$, Lemma PS6.4.2 tells us that $\mathbf{diff}\, u = 0$. Thus $\mathbf{diff}\, x = \mathbf{diff}(u\mathsf{1}t) = \mathbf{diff}\, u + \mathbf{diff}\, \mathsf{1} + \mathbf{diff}\, t = 0 + 0 + \mathbf{diff}\, t = \mathbf{diff}\, t \geq 1$.

$\square$

**Lemma PS6.4.6**
*For all $x, x' \in \Pi_{H,\mathsf{B}}$ and $y, y' \in \Pi_{H,\mathsf{A}}$, if $x\mathsf{0}y = x'\mathsf{0}y'$, then $x = x'$ and $y = y'$.*

**Proof.** Suppose $x, x' \in \Pi_{\mathsf{B}}$, $y, y' \in \Pi_{\mathsf{A}}$, and $x\mathsf{0}y = x'\mathsf{0}y'$.

Suppose, toward a contradiction, that $x \neq x'$. There are two cases to consider.

- Suppose $x$ is a proper prefix of $x'$. Thus $x' = x\mathsf{0}u$ for some $u \in \Sigma^*$. Since $x\mathsf{0}u = x' \in \Pi_{\mathsf{B}}$, Lemma PS6.4.5 tells us that $\mathbf{diff}\, x \geq 1$. But $x \in \Pi_{\mathsf{B}}$, and thus Lemma PS6.4.2 tells us that $\mathbf{diff}\, x = 0$—contradiction.
- Suppose $x'$ is a proper prefix of $x$. This case is similar to the preceding one.

Since we obtained a contradiction in both cases, we have an overall contradiction. Thus $x = x'$.

Since $x = x'$, we have that $x\mathsf{0}y = x'\mathsf{0}y' = x\mathsf{0}y'$, so that $\mathsf{0}y = \mathsf{0}y'$, and thus $y = y'$. $\square$

**Lemma PS6.4.7**
*For all $x, x' \in \Pi_{H,\mathsf{C}}$ and $y, y' \in \Pi_{H,\mathsf{B}}$, if $x\mathsf{1}y = x'\mathsf{1}y'$, then $x = x'$ and $y = y'$.*

**Proof.** Suppose $x, x' \in \Pi_{\mathsf{C}}$, $y, y' \in \Pi_{\mathsf{B}}$, and $x\mathsf{1}y = x'\mathsf{1}y'$.

Suppose, toward a contradiction, that $x \neq x'$. There are two cases to consider.

- Suppose $x$ is a proper prefix of $x'$. Thus $x' = x\mathsf{1}u$ for some $u \in \Sigma^*$. Since $x\mathsf{1}u = x' \in \Pi_{\mathsf{C}}$, Lemma PS6.4.4 tells us that $\mathbf{diff}\, x \geq 1$. But $x \in \Pi_{\mathsf{C}}$, and thus Lemma PS6.4.2 tells us that $\mathbf{diff}\, x = 0$—contradiction.
- Suppose $x'$ is a proper prefix of $x$. This case is similar to the preceding one.

Since we obtained a contradiction in both cases, we have an overall contradiction. Thus $x = x'$.

Since $x = x'$, we have that $x\mathsf{1}y = x'\mathsf{1}y' = x\mathsf{1}y'$, so that $\mathsf{1}y = \mathsf{1}y'$, and thus $y = y'$. $\square$

Finally, the following lemma tells us that $H$ is unambiguous:

**Lemma PS6.4.8**
*For all $pt \in \mathbf{PT}$, for all $pt' \in \mathbf{PT}$, if $pt$ and $pt'$ are valid for $H$, $\mathbf{rootLabel}\, pt = \mathbf{rootLabel}\, pt' \in Q_H$ and $\mathbf{yield}\, pt = \mathbf{yield}\, pt' \in \Sigma^*$, then $pt = pt'$.*

**Proof.** We proceed using the principle of induction on parse trees. There are two parts to show.

(1) Suppose $a \in \mathbf{Sym}$ and $pts \in \mathbf{List}\,\mathbf{PT}$, and assume the inductive hypothesis: for all $i \in [1 : |pts|]$, for all $pt' \in \mathbf{PT}$, if $pts\,i$ and $pt'$ are valid for $H$, $\mathbf{rootLabel}(pts\,i) = \mathbf{rootLabel}\,pt' \in Q_H$ and $\mathbf{yield}(pts\,i) = \mathbf{yield}\,pt' \in \Sigma^*$, then $pts\,i = pt'$. We must show that, for all $pt' \in \mathbf{PT}$, if $(a, pts)$ and $pt'$ are valid for $H$, $\mathbf{rootLabel}(a, pts) = \mathbf{rootLabel}\,pt' \in Q_H$ and $\mathbf{yield}(a, pts) = \mathbf{yield}\,pt' \in \Sigma^*$, then $(a, pts) = pt'$. Suppose $pt' \in \mathbf{PT}$, $(a, pts)$ and $pt'$ are valid for $H$, $\mathbf{rootLabel}(a, pts) = \mathbf{rootLabel}\,pt' \in Q_H$ and $\mathbf{yield}(a, pts) = \mathbf{yield}\,pt' \in \Sigma^*$. We must show that $(a, pts) = pt'$. Since $(a, pts)$ has an element of $Q_H$ as its root label ($a$), and has an element of $\Sigma^*$ as its yield, $pts$ can't be empty. Thus, because $(a, pts)$ is valid for $H$, there are six cases to consider, corresponding to the six productions of $H$.

($\mathsf{A} \to \mathsf{B}$) Suppose $(a, pts) = \mathsf{A}(pt_1)$ for some $pt_1 \in \mathbf{PT}$ such that $pt_1$ is valid for $H$ and $\mathbf{rootLabel}\,pt_1 = \mathsf{B}$. Thus $\mathbf{rootLabel}\,pt' = \mathsf{A}$ and $\mathbf{yield}\,pt_1 = \mathbf{yield}(\mathsf{A}(pt_1)) = \mathbf{yield}\,pt' \in \Sigma^*$. Since $pt'$ is valid for $H$, there are two subcases to consider, corresponding to the two productions of $H$ whose left-hand sides are $\mathsf{A}$.

($\mathsf{A} \to \mathsf{B}$) Suppose $pt' = \mathsf{A}(pt'_1)$ for some $pt'_1 \in \mathbf{PT}$ such that $pt'_1$ is valid for $H$ and $\mathbf{rootLabel}\,pt'_1 = \mathsf{B}$. Since $pt_1$ and $pt'_1$ are valid for $H$, $\mathbf{rootLabel}\,pt_1 = \mathsf{B} = \mathbf{rootLabel}\,pt'_1 \in Q_H$ and $\mathbf{yield}\,pt_1 = \mathbf{yield}\,pt' = \mathbf{yield}(\mathsf{A}(pt'_1)) = \mathbf{yield}\,pt'_1 \in \Sigma^*$, the inductive hypothesis tells us that $pt_1 = pt'_1$. Hence $(a, pts) = \mathsf{A}(pt_1) = \mathsf{A}(pt'_1) = pt'$.

($\mathsf{A} \to \mathsf{B0A}$) Suppose $pt' = \mathsf{A}(pt'_1, pt'_2, pt'_3)$ for some $pt'_1, pt'_2, pt'_3 \in \mathbf{PT}$ such that $pt'_1$, $pt'_2$ and $pt'_3$ are valid for $H$, $\mathbf{rootLabel}\,pt'_1 = \mathsf{B}$, $\mathbf{rootLabel}\,pt'_2 = \mathsf{0}$ and $\mathbf{rootLabel}\,pt'_3 = \mathsf{A}$. Since $pt'_2$ is valid for $H$ and has $\mathsf{0}$ as its root label, it follows that $pt'_2 = \mathsf{0}$, so that $pt' = \mathsf{A}(pt'_1, pt'_2, pt'_3) = \mathsf{A}(pt'_1, \mathsf{0}, pt'_3)$. Thus $\mathbf{yield}\,pt_1 = \mathbf{yield}\,pt' = \mathbf{yield}(\mathsf{A}(pt'_1, \mathsf{0}, pt'_3)) = \mathbf{yield}\,pt'_1\,\mathbf{yield}\,\mathsf{0}\,\mathbf{yield}\,pt'_3 = \mathbf{yield}\,pt'_1\,\mathsf{0}\,\mathbf{yield}\,pt'_3 \in \Sigma^*$. Since $pt_1$ and $pt'_1$ are valid for $H$, $\mathbf{rootLabel}\,pt_1$ and $\mathbf{rootLabel}\,pt'_1$ are $\mathsf{B}$, and $\mathbf{yield}\,pt_1$ and $\mathbf{yield}\,pt'_1$ are in $\Sigma^*$, we have that $\mathbf{yield}\,pt_1$ and $\mathbf{yield}\,pt'_1$ are in $\Pi_{\mathsf{B}}$. Since $\mathbf{yield}\,pt'_1 \in \Pi_{\mathsf{B}}$, Lemma PS6.4.2 tells us that $\mathbf{diff}(\mathbf{yield}\,pt'_1) = 0$. But $\mathbf{yield}\,pt'_1\,\mathsf{0}\,\mathbf{yield}\,pt'_3 = \mathbf{yield}\,pt_1 \in \Pi_{\mathsf{B}}$, and thus Lemma PS6.4.5 tells us that $\mathbf{diff}(\mathbf{yield}\,pt'_1) \geq 1$—contradiction. Thus $(a, pts) = pt'$.

($\mathsf{A} \to \mathsf{B0A}$) Suppose $(a, pts) = \mathsf{A}(pt_1, \mathsf{0}, pt_2)$ for some $pt_1, pt_2 \in \mathbf{PT}$ such that $pt_1$ and $pt_2$ are valid for $H$, $\mathbf{rootLabel}\,pt_1 = \mathsf{B}$ and $\mathbf{rootLabel}\,pt_2 = \mathsf{A}$. Thus $\mathbf{rootLabel}\,pt' = \mathsf{A}$ and $\mathbf{yield}\,pt_1\,\mathsf{0}\,\mathbf{yield}\,pt_2 = \mathbf{yield}(\mathsf{A}(pt_1, \mathsf{0}, pt_2)) = \mathbf{yield}\,pt' \in \Sigma^*$. Since $pt'$ is valid for $H$, there are two subcases to consider.

($\mathsf{A} \to \mathsf{B}$) Suppose $pt' = \mathsf{A}(pt'_1)$ for some $pt'_1 \in \mathbf{PT}$ such that $pt'_1$ is valid for $H$ and $\mathbf{rootLabel}\,pt'_1 = \mathsf{B}$. Thus $\mathbf{yield}\,pt_1\,\mathsf{0}\,\mathbf{yield}\,pt_2 = \mathbf{yield}\,pt' = \mathbf{yield}(\mathsf{A}(pt'_1)) = \mathbf{yield}\,pt'_1 \in \Sigma^*$. Since $\mathbf{yield}\,pt_1 \in \Pi_{\mathsf{B}}$, Lemma PS6.4.2 tells us that $\mathbf{diff}(\mathbf{yield}\,pt_1) = 0$. But $\mathbf{yield}\,pt_1\,\mathsf{0}\,\mathbf{yield}\,pt_2 = \mathbf{yield}\,pt'_1 \in \Pi_{\mathsf{B}}$, and so Lemma PS6.4.5 tells us that $\mathbf{diff}(\mathbf{yield}\,pt_1) \geq 1$—contradiction. Thus $(a, pts) = pt'$.

($\mathsf{A} \to \mathsf{B0A}$) Suppose $pt' = \mathsf{A}(pt'_1, \mathsf{0}, pt'_2)$ for some $pt'_1, pt'_2 \in \mathbf{PT}$ such that $pt'_1$ and $pt'_2$ are valid for $H$, $\mathbf{rootLabel}\,pt'_1 = \mathsf{B}$ and $\mathbf{rootLabel}\,pt'_2 = \mathsf{A}$. Thus $\mathbf{yield}\,pt_1\,\mathsf{0}\,\mathbf{yield}\,pt_2 = \mathbf{yield}\,pt' = \mathbf{yield}(\mathsf{A}(pt'_1, \mathsf{0}, pt'_2)) = \mathbf{yield}\,pt'_1\,\mathsf{0}\,\mathbf{yield}\,pt'_2 \in \Sigma^*$. Since $\mathbf{yield}\,pt_1, \mathbf{yield}\,pt'_1 \in \Pi_{\mathsf{B}}$ and $\mathbf{yield}\,pt_2, \mathbf{yield}\,pt'_2 \in \Pi_{\mathsf{A}}$, Lemma PS6.4.6

tells us that $\textbf{yield}\,pt_1 = \textbf{yield}\,pt'_1 \in \Sigma^*$ and $\textbf{yield}\,pt_2 = \textbf{yield}\,pt'_2 \in \Sigma^*$. Thus, by the inductive hypothesis, we have that $pt_1 = pt'_1$ and $pt_2 = pt'_2$, so that $(a, pts) = \mathsf{A}(pt_1, \mathsf{0}, pt_2) = \mathsf{A}(pt'_1, \mathsf{0}, pt'_2) = pt'$.

($\mathsf{B} \to \mathsf{C}$) Suppose $(a, pts) = \mathsf{B}(pt_1)$ for some $pt_1 \in \textbf{PT}$ such that $pt_1$ is valid for $H$ and $\textbf{rootLabel}\,pt_1 = \mathsf{C}$. Thus $\textbf{rootLabel}\,pt' = \mathsf{B}$ and $\textbf{yield}\,pt_1 = \textbf{yield}(\mathsf{B}(pt_1)) = \textbf{yield}\,pt' \in \Sigma^*$. Since $pt'$ is valid for $H$, there are two subcases to consider.

($\mathsf{B} \to \mathsf{C}$) Suppose $pt' = \mathsf{B}(pt'_1)$ for some $pt'_1 \in \textbf{PT}$ such that $pt'_1$ is valid for $H$ and $\textbf{rootLabel}\,pt'_1 = \mathsf{C}$. Thus $\textbf{yield}\,pt_1 = \textbf{yield}\,pt' = \textbf{yield}(\mathsf{B}(pt'_1)) = \textbf{yield}\,pt'_1 \in \Sigma^*$. Hence, by the inductive hypothesis, we have that $pt_1 = pt'_1$, so that $(a, pts) = \mathsf{B}(pt_1) = \mathsf{B}(pt'_1) = pt'$.

($\mathsf{B} \to \mathsf{C1B}$) Suppose $pt' = \mathsf{B}(pt'_1, \mathsf{1}, pt'_2)$ for some $pt'_1, pt'_2 \in \textbf{PT}$ such that $pt'_1$ and $pt'_2$ are valid for $H$, $\textbf{rootLabel}\,pt'_1 = \mathsf{C}$ and $\textbf{rootLabel}\,pt'_2 = \mathsf{B}$. Thus $\textbf{yield}\,pt_1 = \textbf{yield}\,pt' = \textbf{yield}(\mathsf{B}(pt'_1, \mathsf{1}, pt'_2)) = \textbf{yield}\,pt'_1\,\mathsf{1}\,\textbf{yield}\,pt'_2 \in \Sigma^*$. Since $\textbf{yield}\,pt'_1 \in \Pi_{\mathsf{C}}$, Lemma PS6.4.2 tells us that $\textbf{diff}(\textbf{yield}\,pt'_1) = 0$. But $\textbf{yield}\,pt'_1\,\mathsf{1}\,\textbf{yield}\,pt'_2 = \textbf{yield}\,pt_1 \in \Pi_{\mathsf{C}}$, and thus Lemma PS6.4.4 tells us that $\textbf{diff}(\textbf{yield}\,pt'_1) \geq 1$—contradiction. Thus $(a, pts) = pt'$.

($\mathsf{B} \to \mathsf{C1B}$) Suppose $(a, pts) = \mathsf{B}(pt_1, \mathsf{1}, pt_2)$ for some $pt_1, pt_2 \in \textbf{PT}$ such that $pt_1$ and $pt_2$ are valid for $H$, $\textbf{rootLabel}\,pt_1 = \mathsf{C}$ and $\textbf{rootLabel}\,pt_2 = \mathsf{B}$. Thus $\textbf{rootLabel}\,pt' = \mathsf{B}$ and $\textbf{yield}\,pt_1\,\mathsf{1}\,\textbf{yield}\,pt_2 = \textbf{yield}(\mathsf{B}(pt_1, \mathsf{1}, pt_2)) = \textbf{yield}\,pt' \in \Sigma^*$. Since $pt'$ is valid for $H$, there are two subcases to consider.

($\mathsf{B} \to \mathsf{C}$) Suppose $pt' = \mathsf{B}(pt'_1)$ for some $pt'_1 \in \textbf{PT}$ such that $pt'_1$ is valid for $H$ and $\textbf{rootLabel}\,pt'_1 = \mathsf{C}$. Thus $\textbf{yield}\,pt_1\,\mathsf{1}\,\textbf{yield}\,pt_2 = \textbf{yield}\,pt' = \textbf{yield}(\mathsf{B}(pt'_1) = \textbf{yield}\,pt'_1 \in \Sigma^*$. Since $\textbf{yield}\,pt_1 \in \Pi_{\mathsf{C}}$, Lemma PS6.4.2 tells us that $\textbf{diff}(\textbf{yield}\,pt_1) = 0$. But $\textbf{yield}\,pt_1\,\mathsf{1}\,\textbf{yield}\,pt_2 = \textbf{yield}\,pt'_1 \in \Pi_{\mathsf{C}}$, and so Lemma PS6.4.4 tells us that $\textbf{diff}(\textbf{yield}\,pt_1) \geq 1$—contradiction. Thus $(a, pts) = pt'$.

($\mathsf{B} \to \mathsf{C1B}$) Suppose $pt' = \mathsf{B}(pt'_1, \mathsf{1}, pt'_2)$ for some $pt'_1, pt'_2 \in \textbf{PT}$ such that $pt'_1$ and $pt'_2$ are valid for $H$, $\textbf{rootLabel}\,pt'_1 = \mathsf{C}$ and $\textbf{rootLabel}\,pt'_2 = \mathsf{B}$. Thus $\textbf{yield}\,pt_1\,\mathsf{1}\,\textbf{yield}\,pt_2 = \textbf{yield}\,pt' = \textbf{yield}(\mathsf{B}(pt'_1, \mathsf{1}, pt'_2)) = \textbf{yield}\,pt'_1\,\mathsf{1}\,\textbf{yield}\,pt'_2 \in \Sigma^*$. Since $\textbf{yield}\,pt_1, \textbf{yield}\,pt'_1 \in \Pi_{\mathsf{C}}$ and $\textbf{yield}\,pt_2, \textbf{yield}\,pt'_2 \in \Pi_{\mathsf{B}}$, Lemma PS6.4.7 tells us that $\textbf{yield}\,pt_1 = \textbf{yield}\,pt'_1 \in \Sigma^*$ and $\textbf{yield}\,pt_2 = \textbf{yield}\,pt'_2 \in \Sigma^*$. Thus, by the inductive hypothesis, we have that $pt_1 = pt'_1$ and $pt_2 = pt'_2$, so that $(a, pts) = \mathsf{B}(pt_1, \mathsf{1}, pt_2) = \mathsf{B}(pt'_1, \mathsf{1}, pt'_2) = pt'$.

($\mathsf{C} \to \mathsf{2}$) Suppose $(a, pts) = \mathsf{C}(\mathsf{2})$. Thus $\textbf{rootLabel}\,pt' = \mathsf{C}$ and $\mathsf{2} = \textbf{yield}(\mathsf{C}(\mathsf{2})) = \textbf{yield}\,pt' \in \Sigma^*$. Since $pt'$ is valid for $H$, there are two subcases to consider.

($\mathsf{C} \to \mathsf{2}$) Suppose $pt' = \mathsf{C}(\mathsf{2})$. Then $(a, pts) = \mathsf{C}(\mathsf{2}) = pt'$.

($\mathsf{C} \to \mathsf{3A4}$) Suppose $pt' = \mathsf{C}(\mathsf{3}, pt'_1, \mathsf{4})$ for some $pt'_1 \in \textbf{PT}$ such that $pt'_1$ is valid for $H$ and $\textbf{rootLabel}\,pt'_1 = \mathsf{A}$. Thus $\mathsf{2} = \textbf{yield}\,pt' = \textbf{yield}(\mathsf{C}(\mathsf{3}, pt'_1, \mathsf{4})) = \mathsf{3}\,\textbf{yield}\,pt'_1\,\mathsf{4}$—contradiction. Thus $(a, pts) = pt'$.

($\mathsf{C} \to \mathsf{3A4}$) Suppose $(a, pts) = \mathsf{C}(\mathsf{3}, pt_1, \mathsf{4})$ for some $pt_1 \in \textbf{PT}$ such that $pt_1$ is valid for $H$ and $\textbf{rootLabel}\,pt_1 = \mathsf{A}$. Thus $\textbf{rootLabel}\,pt' = \mathsf{C}$ and $\mathsf{3}\,\textbf{yield}\,pt_1\,\mathsf{4} = \textbf{yield}(\mathsf{C}(\mathsf{3}, pt_1, \mathsf{4})) = \textbf{yield}\,pt' \in \Sigma^*$. Since $pt'$ is valid for $H$, there are two subcases to consider.

($\mathsf{C} \to \mathsf{2}$) Suppose $pt' = \mathsf{C}(\mathsf{2})$. Then $\mathsf{3}\,\mathbf{yield}\,pt_1\,\mathsf{4} = \mathbf{yield}\,pt' = \mathbf{yield}(\mathsf{C}(\mathsf{2})) = \mathsf{2}$—contradiction. Thus $(a, pts) = pt'$.

($\mathsf{C} \to \mathsf{3A4}$) Suppose $pt' = \mathsf{C}(\mathsf{3}, pt'_1, \mathsf{4})$ for some $pt'_1 \in \mathbf{PT}$ such that $pt'_1$ is valid for $H$ and $\mathbf{rootLabel}\,pt'_1 = \mathsf{A}$. Thus $\mathsf{3}\,\mathbf{yield}\,pt_1\,\mathsf{4} = \mathbf{yield}\,pt' = \mathbf{yield}(\mathsf{C}(\mathsf{3}, pt'_1, \mathsf{4})) = \mathsf{3}\,\mathbf{yield}\,pt'_1\,\mathsf{4} \in \Sigma^*$, so that $\mathbf{yield}\,pt_1 = \mathbf{yield}\,pt'_1 \in \Sigma^*$. Hence, by the inductive hypothesis, we have that $pt_1 = pt'_1$, so that $(a, pts) = \mathsf{C}(\mathsf{3}, pt_1, \mathsf{4}) = \mathsf{C}(\mathsf{3}, pt'_1, \mathsf{4}) = pt'$.

(2) Suppose $a \in \mathbf{Sym}$. We must show that, for all $pt' \in \mathbf{PT}$, if $a(\%)$ and $pt'$ are valid for $H$, $\mathbf{rootLabel}(a(\%)) = \mathbf{rootLabel}\,pt' \in Q_H$ and $\mathbf{yield}(a(\%)) = \mathbf{yield}\,pt' \in \Sigma^*$, then $a(\%) = pt'$. And this follows since $a(\%)$ is not valid for $H$ (because $H$ has no %-productions).

$\square$