Chapter 1: Mathematical Background

This chapter consists of the material on set theory, induction, inductive definitions and recursion that will be required in later chapters.

1.1: Basic Set Theory

In this section, we will cover the material on sets, relations, functions and data structures that will be needed in what follows.

Much of this material should be at least partly familiar.

Describing Sets by Listing Their Elements; Sets of Numbers

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We write:

- $\mathbb{N}$ for the set $\{\quad 1, \ldots\}$ of all natural numbers;
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Relationships between Sets

Sets A and B are equal ($A = B$) iff (if and only if) they have the same elements, i.e., for all x, $x \in A$ iff $x \in B$.

Suppose A and B are sets. We say that:

- A is a *subset* of B ($A \subseteq B$) iff, for all $x \in A$, $x \in B$;
- A is a *proper subset* of B ($A \subsetneq B$) iff $A \subseteq B$ but $A \neq B$.

For example, $\emptyset$ $\quad$ $\mathbb{N}$, $\mathbb{N}$ $\quad$ $\mathbb{N}$ and $\mathbb{N}$ $\quad$ $\mathbb{Z}$.

We also have the notions of superset ($A \supseteq B$) and proper superset ($A \supsetneq B$).

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Set Formation

We will make extensive use of the $\{ \cdots \mid \cdots \}$ notation for forming sets. Let's consider two representative examples of its use.

Let

$$A = \{ n \mid n \in \mathbb{N} \text{ and } n^2 \geq 20 \} = \{ n \in \mathbb{N} \mid n^2 \geq 20 \}.$$

Then, for all n,

$$n \in A \quad \text{iff} \quad n \in \mathbb{N} \text{ and } n^2 \geq 20.$$

Is $5 \in A$?

Is $5.5 \in A$?

Is $4 \in A$?

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Is $4 \in A$? No—$4^2 \not\geq 20$.

Set Formation (Cont.)

Let

$$B = \{\, n^3 + m^2 \mid n, m \in \mathbb{N} \text{ and } n, m \geq 1 \,\}.$$

Then, for all l,

$$\begin{aligned} l \in B \quad &\text{iff} \quad l = n^3 + m^2, \text{ for some } n, m \text{ such that } n, m \in \mathbb{N} \text{ and } n, m \geq 1 \\ &\text{iff} \quad l = n^3 + m^2, \text{ for some } n, m \in \mathbb{N} \text{ such that } n, m \geq 1. \end{aligned}$$

Is $9 \in B$?

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Is $9 \in B$? To answer "yes", we must show

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for some values of n, m.

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Is $9 \in B$? To answer "yes", we must show

$$9 = n^3 + m^2 \text{ and } n, m \in \mathbb{N} \text{ and } n, m \geq 1,$$

for some values of n, m. Yes—$9 = 2^3 + 1^2$ and $2, 1 \in \mathbb{N}$ and $2, 1 \geq 1$.

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Given $n, m \in \mathbb{Z}$, we write $[n : m]$ for $\{\, l \in \mathbb{Z} \mid l \geq n \text{ and } l \leq m \,\}$.

Thus $[n : m]$ is all of the integers that are at least n and no more than m.

For example, $[-2 : 1]$ is $\{-2, -1, 0, 1\}$ and $[3 : 2]$ is

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For example, $[-2 : 1]$ is $\{-2, -1, 0, 1\}$ and $[3 : 2]$ is $\emptyset$.

Operations on Sets

Recall the following operations on sets A and B:

$$A \cup B = \{\, x \mid x \in A \text{ or } x \in B \,\} \qquad \text{(union)}$$
$$A \cap B = \{\, x \mid x \in A \text{ and } x \in B \,\} \qquad \text{(intersection)}$$
$$A - B = \{\, x \in A \mid x \notin B \,\} \qquad \text{(difference)}$$
$$A \times B = \{\, (x, y) \mid x \in A \text{ and } y \in B \,\} \qquad \text{(product)}$$
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Generalized Union and Intersection

If X is a set of sets, then the *generalized union* of X ($\bigcup X$) is

$$\{\, a \mid a \in A, \text{for some } A \in X \,\}.$$

For example

$$\bigcup\{\{0,1\},\{1,2\},\{2,3\}\} =$$

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If X is a *nonempty* set of sets, then the *generalized intersection* of X ($\bigcap X$) is

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The *domain* of a relation R ($\textbf{domain}\, R$) is $\{\, x \mid (x, y) \in R,$ for some $y \,\}$, and the *range* of R ($\textbf{range}\, R$) is $\{\, y \mid (x, y) \in R,$ for some $x \,\}$.

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We often write $x\ R\ y$ for $(x, y) \in R$.

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Properties of Relations

A relation R is:

- *reflexive on* a set A iff, for all $x \in A$, $(x, x) \in R$;
- *transitive* iff, for all x, y, z, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$;
- *symmetric* iff, for all x, y, if $(x, y) \in R$, then $(y, x) \in R$;
- a *function* iff, for all x, y, z, if $(x, y) \in R$ and $(x, z) \in R$, then $y = z$.

Is $R = \{(0, 1), (1, 2), (0, 2)\}$ reflexive on $\{0, 1, 2\}$?
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Is R transitive? Yes; since $(0, 1), (1, 2) \in R$, $(0, 2) \in R$ required.
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Properties of Relations

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- *symmetric* iff, for all x, y, if $(x, y) \in R$, then $(y, x) \in R$;
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The book talks about *total orderings* like $\leq$ on $\mathbb{N}$, as well as the corresponding *strict total orderings*, like $<$ on $\mathbb{N}$.

More on Functions

The relation

$$f = \{(0,1),(1,2),(2,0)\}$$

is a function.

If f is a function and $x \in \textbf{domain}\, f$, we write $f\, x$ for the *application* of f to x, i.e., the unique y such that $(x, y) \in f$.

We say that f is a *function from* a set X *to* a set Y iff f is a function, $\textbf{domain}\, f = X$ and $\textbf{range}\, f \subseteq Y$.

We write $X \to Y$ for the set of all functions from X to Y.

For the f defined above, we have that $f\,0 =$, $f\,1 =$, $f\,2 =$, f is a function from to , and $f \in \quad \to$

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Bijections

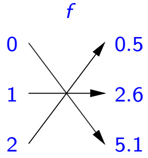
A *bijection* f *from* a set X *to* a set Y is a function from X to Y such that, for all $y \in Y$, there is a unique $x \in X$ such that $(x, y) \in f$.

For example,

$$f = \{(0, 5.1), (1, 2.6), (2, 0.5)\}$$

is a bijection from $\{0, 1, 2\}$ to $\{0.5, 2.6, 5.1\}$.

We can visualize f as a one-to-one correspondence between these sets:



Set Cardinality

We say that a set X has the *same size* as a set Y ($X \cong Y$) iff there is a bijection from X to Y. It's not hard to show that for all sets X, Y, Z:

- $X \cong X$;
- If $X \cong Y \cong Z$, then $X \cong Z$;
- If $X \cong Y$, then $Y \cong X$.

Finite and Infinite Sets

A set X is *finite* iff $X \cong [1:n]$, for some $n \in \mathbb{N}$; otherwise X is *infinite*.

A set X is *countably infinite* iff $X \cong \mathbb{N}$.

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Every set X has a *size* or *cardinality* ($|X|$) and we have that, for all sets X and Y, $|X| = |Y|$ iff $X \cong Y$. The sizes of finite sets are natural numbers.

Set Size Examples

- The sets $\emptyset$ and $\{0.5, 2.6, 5.1\}$ are finite, and are thus also countable;
- The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathcal{P}\,\mathbb{N}$ are infinite;
- The set $\mathbb{N}$ is countably infinite, and is thus countable;
- The set $\mathbb{Z}$ is countably infinite, and is thus countable, because of the existence of the following bijection:

$\cdots$	-2	-1	0	1	2	$\cdots$
$\cdots$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$
$\cdots$		2	0	1		$\cdots$

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$$\begin{array}{ccccccc} \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\ \cdots & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\ \cdots & 4 & 2 & 0 & 1 & 3 & \cdots \end{array}$$

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Data Structures: Booleans

$\textbf{Bool} = \{\textbf{true}, \textbf{false}\}$.

We have the usual negation (**not**), conjunction (**and**) and disjunction (**or**) operations on booleans.

Options

$\textbf{Option}\, X = \{\textbf{none}\} \cup \{\, \textbf{some}\, x \mid x \in X \,\}$.

For example, $\textbf{Option Bool} = \{\textbf{none}, \textbf{some true}, \textbf{some false}\}$.

E.g., we could define a function $f \in \mathbb{N} \times \mathbb{N} \to \textbf{Option Bool}$ by:

$$f(n, m) = \begin{cases} \textbf{none}, & \text{if } m = 0, \\ \textbf{some true} & \text{if } m \neq 0 \text{ and } n = ml \text{ for some } l \in \mathbb{N}, \\ \textbf{some false} & \text{if } m \neq 0 \text{ and } n \neq ml \text{ for all } l \in \mathbb{N}. \end{cases}$$

Lists

A *list* is a function with domain $[1:n]$, for some $n \in \mathbb{N}$.

For example $\emptyset$ is a list, as it is a function with domain

And $\{(1,3),(2,5),(3,7)\}$ is a list, as it is a function with domain

We abbreviate a list $\{(1,x_1),(2,x_2),\ldots,(n,x_n)\}$ to $[x_1,x_2,\ldots,x_n]$.
Thus $\emptyset$ and $\{(1,3),(2,5),(3,7)\}$ are abbreviated to $[\,]$ and $[3,5,7]$.

$|\cdot|$ doubles as list

$f \mathbin{@} g$ is list concatenation. E.g., $[2,3,4] \mathbin{@} [5,6] = [2,3,4,5,6]$.

Concatenation is associative $(f \mathbin{@} g) \mathbin{@} h = f \mathbin{@} (g \mathbin{@} h)$ and has $[\,]$ as its identity ($[\,] \mathbin{@} f = f = f \mathbin{@} [\,]$).

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