3.14: The Pumping Lemma for Regular Languages

In this section we consider techniques for showing that particular languages are not regular.

Introduction

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Answer: No.

Intuitively, an automaton would have to have infinitely many states to accept this language. A finite automaton won't be able to keep track of how many 0's it has seen so far, and thus won't be able to insist that the correct number of 1's follow.

Introduction

We could turn the preceding ideas into a direct proof that L is not regular.

Instead, we will first state a general result, called the Pumping Lemma for regular languages, for proving that languages are non-regular.

Next, we will show how the Pumping Lemma can be used to prove that L is non-regular.

Finally, we will prove the Pumping Lemma.

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(3) $uv^i w \in L$, *for all* $i \in \mathbb{N}$.

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$$u = 0^i, \quad v = 0^j, \quad w = 0^k 1^n, \quad i + j + k = n.$$

By (2), we have that

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$$0^{i+k} 1^n = 0^i 0^k 1^n = uw = u\%w = uv^0 w \in L.$$

Thus $i + k = n$—contradiction. Thus L is not regular.

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$$q_1 \overset{a_1}{\Rightarrow} q_2 \overset{a_2}{\Rightarrow} \cdots q_m \overset{a_m}{\Rightarrow} q_{m+1},$$

that is labeled by z and where $q_1 = s_M$, $q_{m+1} \in A_M$ and $a_i \in$ for all $1 \leq i \leq m$.

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that is labeled by z and where $q_1 = s_M$, $q_{m+1} \in A_M$ and $a_i \in \textbf{Sym}$ for all $1 \leq i \leq m$. Since $|Q_M| = n$, not all of the states $q_1, \ldots, q_{n+1}$ are distinct. Thus, there are $1 \leq i < j \leq n+1$ such that $q_i = q_j$.

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Let

$$u = a_1 \cdots a_{i-1}, \quad v = a_i \cdots a_{j-1}, \quad w = a_j \cdots a_m.$$

Then $z = uvw$. Since $|uv| = j - 1$ and $j \leq n + 1$, we have that $|uv| \leq n$. Since $i < j$, we have that $i \leq j - 1$, and thus that $v \neq \%$.

Proof (cont.). Finally, since

$$q_i \in \Delta(\{q_1\}, u), \quad q_j \in \Delta(\{q_i\}, v), \quad q_{m+1} \in \Delta(\{q_j\}, w)$$

and $q_i = q_j$, we have that

$$q_j \in \Delta(\{q_1\}, u), \quad q_j \in \Delta(\{q_j\}, v), \quad q_{m+1} \in \Delta(\{q_j\}, w).$$

Thus, we have that $q_{m+1} \in \Delta(\{q_1\}, uv^i w)$ for all $i \in \mathbb{N}$. But $q_1 = s_M$ and $q_{m+1} \in A_M$, and thus $uv^i w \in L(M) = L$ for all $i \in \mathbb{N}$. □

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Suppose, toward a contradiction, that L' is regular. It is easy to see that $\{0\}$ and $\{1\}$ are regular (e.g., they are generated by the regular expressions 0 and 1). Thus, by Theorem 3.12.17, we have that $\{0\}^*\{1\}^*$ is regular. Hence, by Theorem 3.12.17 again, it follows that —contradiction. Thus L' is non-regular.

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Forlan Implementation of Pumping Lemma

The textbook describes the implementation in Forlan of the idea behind the Pumping Lemma.