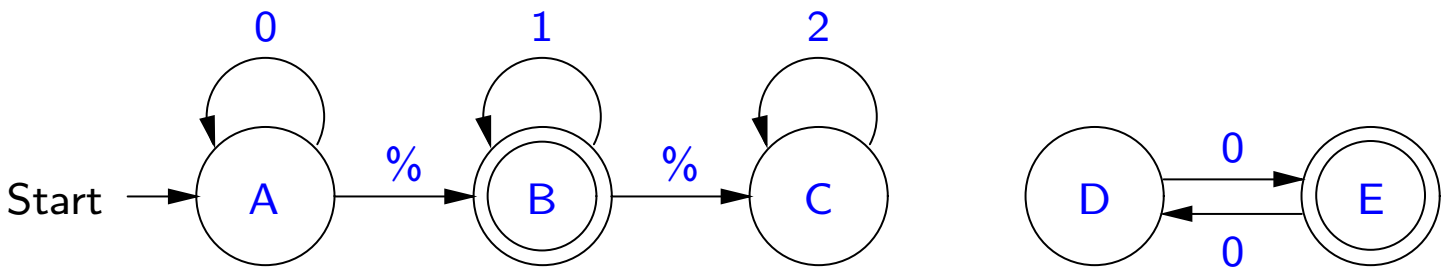
# 3.7: Simplification of Finite Automata

In this section, we: say what it means for a finite automaton to be simplified; study an algorithm for simplifying finite automata; and see how finite automata can be simplified in Forlan.

Suppose $M$ is the finite automaton




What is odd about $M$?

First, there are no valid labeled paths from the start state to D and E, and so these states are redundant.

Second, there are no valid labeled paths from C to an accepting state, and so it is also redundant.

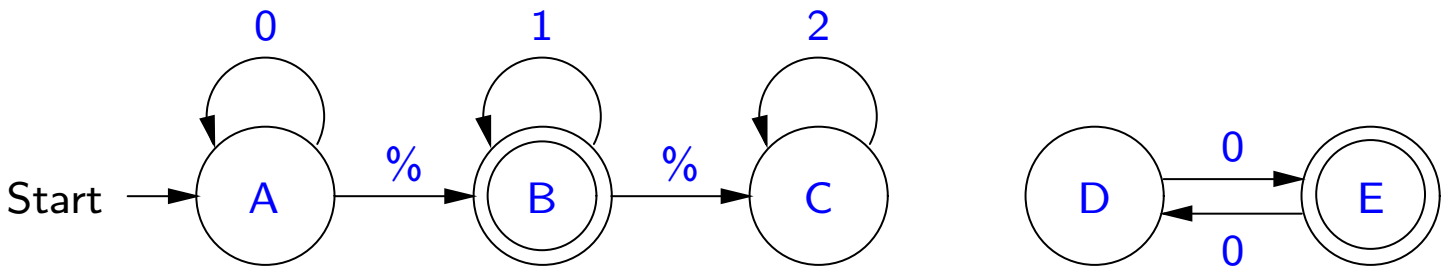
# *Useful States*

Suppose $M$ is a finite automaton. We say that a state $q \in Q_M$ is:

- *reachable in* $M$ iff there is a labeled path $lp$ such that $lp$ is valid for $M$, the start state of $lp$ is $s_M$, and the end state of $lp$ is $q$;
- *live in* $M$ iff there is a labeled path $lp$ such that $lp$ is valid for $M$, the start state of $lp$ is $q$, and the end state of $lp$ is in $A_M$;
- *dead in* $M$ iff $q$ is not live in $M$; and
- *useful in* $M$ iff $q$ is both reachable and live in $M$.

# *Useful States Example*

Let $M$ be our example finite automaton:




The reachable states of $M$ are: A, B and C. The live states of $M$ are: A, B, D and E. And, the useful states of $M$ are: A and B.

# *Generating Reachable, Live and Useful States*

There is a simple algorithm for generating the set of reachable states of a finite automaton $M$. We generate the least subset $X$ of $Q_M$ such that:

- $s_M \in X$; and
- for all $q, r \in Q_M$ and $x \in \mathbf{Str}$, if $q \in X$ and $(q, x, r) \in T_M$, then $r \in X$.

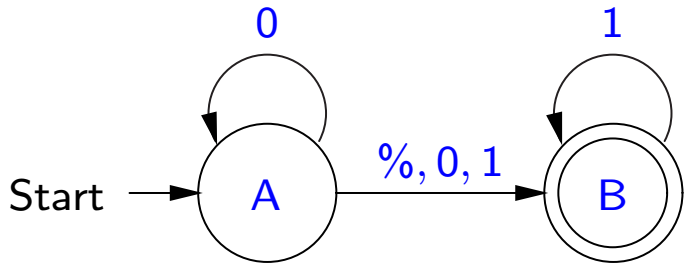
Similarly, there is a simple algorithm for generating the set of live states of a finite automaton $M$. We generate the least subset $Y$ of $Q_M$ such that:

- $A_M \subseteq Y$; and
- for all $q, r \in Q_M$ and $x \in \mathbf{Str}$, if $r \in Y$ and $(q, x, r) \in T_M$, then $q \in Y$.

Thus, we can generate the set of useful states of an FA by generating the set of reachable states, generating the set of live states, and intersecting those sets of states.

# *Redundant Transitions*

Now, suppose $N$ is the FA




What is odd about this machine?

Here, the transitions $(\mathsf{A}, 0, \mathsf{B})$ and $(\mathsf{A}, 1, \mathsf{B})$ are redundant.

Given an FA $M$ and a finite subset $U$ of $\{\, (q, x, r) \mid q, r \in Q_M$ and $x \in \mathbf{Str}\,\}$, we write $M/U$ for the FA that is identical to $M$ except that its set of transitions is $U$.

If $M$ is an FA and $(p, x, q) \in T_M$, we say that:

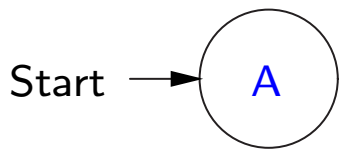
- $(p, x, q)$ *is redundant in* $M$ iff $q \in \Delta_N(\{p\}, x)$, where $N = M/(T_M - \{(p, x, q)\})$; and
- $(p, x, q)$ *is irredundant in* $M$ iff $(p, x, q)$ is not redundant in $M$.

# *Definition of Simplification*

We say that a finite automaton $M$ is *simplified* iff either

- every state of $M$ is useful, and every transition of $M$ is irredundant; or
- $|Q_M| = 1$ and $A_M = T_M = \emptyset$.

Thus the FA




is simplified, even though its start state is not live, and is thus not useful.

**Proposition 3.7.1**
*If $M$ is a simplified finite automaton, then*
**alphabet** $M =$ **alphabet**$(L(M))$.

# *Algorithm for Removing Redundant Transitions*

Given an FA $M$, $p, q \in Q_M$ and $x \in \mathbf{Str}$, we say that $(p, x, q)$ *is implicit in* $M$ iff $q \in \Delta_M(\{p\}, x)$.

Given an FA $M$, we define a function $\mathbf{remRedun}_M \in \mathcal{P}\, T_M \times \mathcal{P}\, T_M \to \mathcal{P}\, T_M$ by well-founded recursion on the size of its second argument.

For $U, V \subseteq T_M$, $\mathbf{remRedun}(U, V)$ proceeds as follows:

- If $V = \emptyset$, then it returns $U$.
- Otherwise, let $v$ be the greatest element of $V$, and $V' = V - \{v\}$. If $v$ is implicit in $M/(U \cup V')$, then **remRedun** returns the result of evaluating $\mathbf{remRedun}(U, V')$. Otherwise, it returns the result of evaluating $\mathbf{remRedun}(U \cup \{v\}, V')$.

In general, there are multiple—incompatible—ways of removing redundant transitions from an FA. **remRedun** is defined so as to favor removing transitions that are larger in our total ordering on transitions.

# *Simplification Algorithm*

We define a function **simplify** $\in$ **FA** $\to$ **FA** by: **simplify** $M$ is the finite automaton $N$ produced by the following process.

- First, the useful states are $M$ are determined.
- If $s_M$ is not useful in $M$, the $N$ is defined by:
  - $Q_N = \{s_M\}$;
  - $s_N = s_M$;
  - $A_N = \emptyset$; and
  - $T_N = \emptyset$.
- And, if $s_M$ is useful in $M$, then $N$ is defined by:
  - $Q_N = \{\, q \in Q_M \mid q \text{ is useful in } M \,\}$;
  - $s_N = s_M$;
  - $A_N = A_M \cap Q_N = \{\, q \in A_M \mid q \in Q_N \,\}$; and
  - $T_N = \textbf{remRedun}(\emptyset, \{\, (q, x, r) \in T_M \mid q, r \in Q_N \,\})$.

# *More on Simplification Algorithm*

**Proposition 3.7.3**

*Suppose $M$ is a finite automaton. Then:*

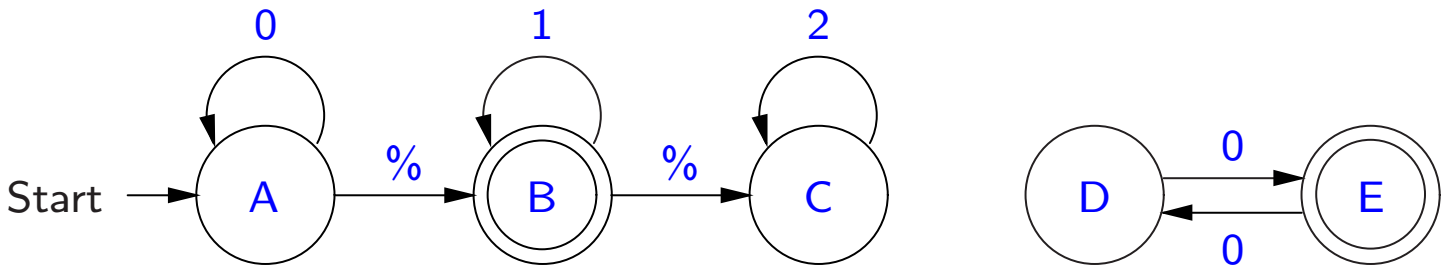
*(1)* **simplify** $M$ *is simplified;*

*(2)* **simplify** $M \approx M$*; and*

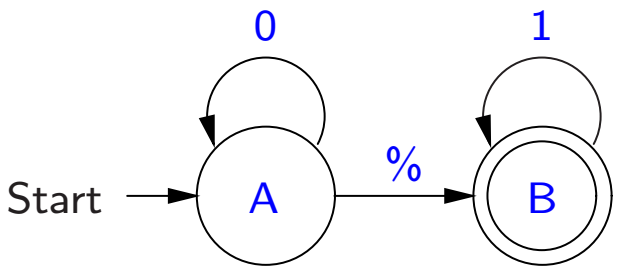
*(3)* **alphabet**(**simplify** $M$) = **alphabet**$(L(M)) \subseteq$ **alphabet** $M$.

# *Simplification Examples*

If $M$ is the finite automaton

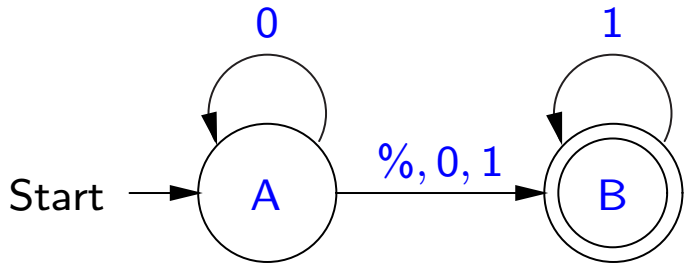



then **simplify** $M$ is the finite automaton

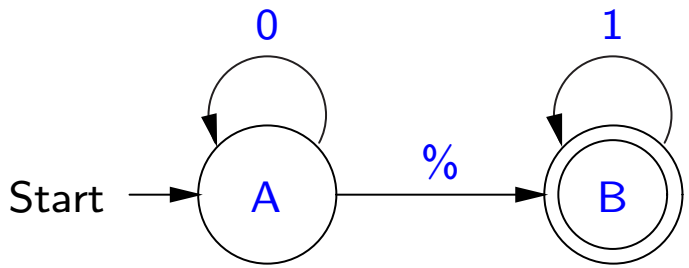


# *Simplification Examples*

If $N$ is the finite automaton




then **simplify** $N$ is the finite automaton



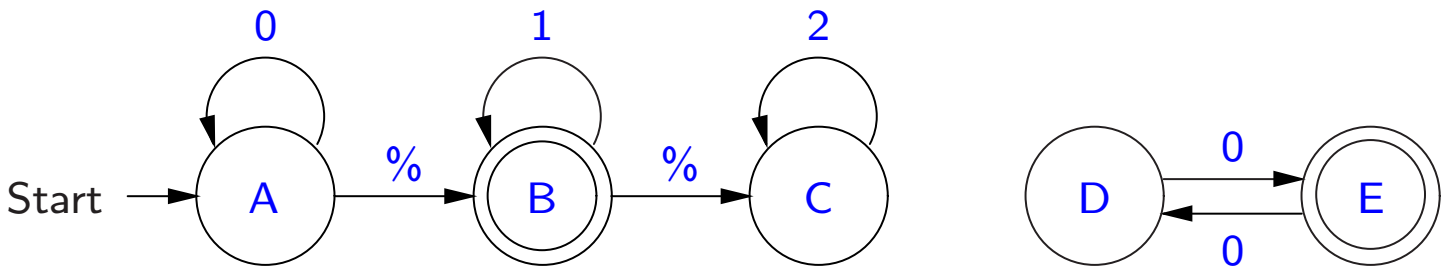
# *Simplification in Forlan*

The Forlan module `FA` includes the following functions relating to the simplification of finite automata:

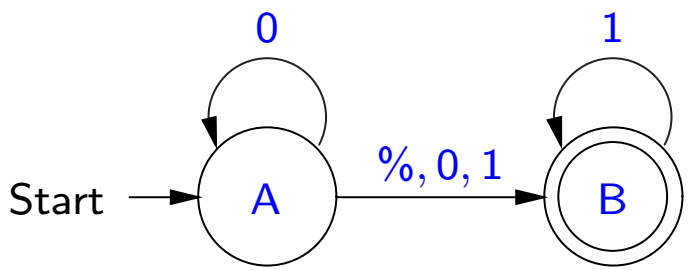
```
val simplify   : fa -> fa
val simplified : fa -> bool
```

# *Simplication Examples in Forlan*

In the following, suppose `fa1` is the finite automaton

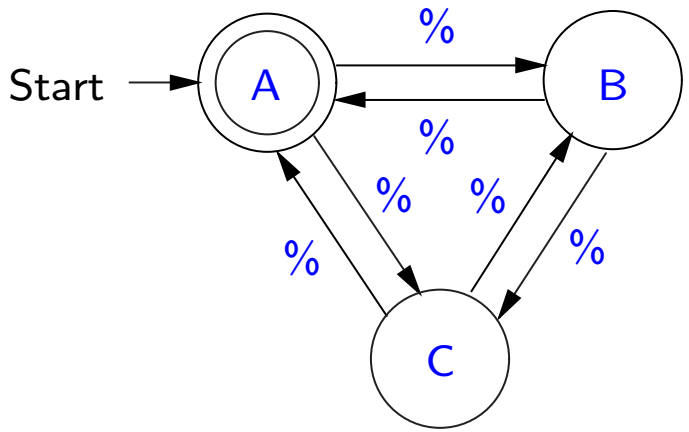



`fa2` is the finite automaton



# *Simplication Examples in Forlan*

and `fa3` is the finita automaton



# *Simplication Examples in Forlan*

Here are some example uses of `simplify` and `simplified`:

```
- FA.simplified fa1;
val it = false : bool
- val fa1' = FA.simplify fa1;
val fa1' = - : fa
- FA.output("", fa1');
{states} A, B {start state} A {accepting states} B
{transitions} A, % -> B; A, 0 -> A; B, 1 -> B
val it = () : unit
- FA.simplified fa1';
val it = true : bool
- val fa2' = FA.simplify fa2;
val fa2' = - : fa
- FA.output("", fa2');
{states} A, B {start state} A {accepting states} B
{transitions} A, % -> B; A, 0 -> A; B, 1 -> B
val it = () : unit
```

# *Simplication Examples in Forlan*

```
- val fa3' = FA.simplify fa3;
val fa3' = - : fa
- FA.output("", fa3');
{states} A, B, C {start state} A {accepting states} A
{transitions} A, % -> B | C; B, % -> A; C, % -> A
val it = () : unit
```

Thus the simplification of `fa3` resulted in the removal of the %-transitions between B and C.