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- intersecting a grammar and an empty-string finite automaton; and
- subtracting a deterministic finite automaton from a grammar.

Thus, if L_1 is a context-free language, and L_2 is a regular language, we will have that $L_1 \cap L_2$ and $L_1 - L_2$ are context-free.

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The book shows several additional closure properties of context-free languages, in addition to giving the corresponding operations on grammars.

Basic Grammars and Operations on Grammars

The grammar, **emptyStr**, with variable A and production $A \rightarrow \%$ generates the language $\{\%\}$.

The grammar, **emptySet**, with variable A and no productions generates the language \emptyset .

If w is a string, then the grammar with variable A and production $A \rightarrow w$ generates the language $\{w\}$.

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If w is a string, then the grammar with variable A and production $A \rightarrow w$ generates the language $\{w\}$. Actually, we must be careful to choose a variable that doesn't occur in w . We can do that by adding as many nested \langle and \rangle around A as needed (A , $\langle A \rangle$, $\langle \langle A \rangle \rangle$, etc.).

This defines functions **strToGram** $\in \mathbf{Str} \rightarrow \mathbf{Gram}$ and **symToGram** $\in \mathbf{Sym} \rightarrow \mathbf{Gram}$.

Union Operation

Suppose G_1 and G_2 are grammars. We can define a grammar H such that $L(H) = L(G_1) \cup L(G_2)$ by unioning together the variables and productions of G_1 and G_2 , and adding a new start variable q , along with productions

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$$q \rightarrow s_{G_1} \mid s_{G_2}.$$

What do we have to know about G_1 , G_2 and q for the above to be valid?

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- $Q_{G_1} \cap Q_{G_2} = \emptyset$ and $q \notin Q_{G_1} \cup Q_{G_2}$; and
- **alphabet** $G_1 \cap Q_{G_2} = \emptyset$, **alphabet** $G_2 \cap Q_{G_1} = \emptyset$ and $q \notin$ **alphabet** $G_1 \cup$ **alphabet** G_2 .

Union Operation

Our official union operation for grammars renames the variables of G_1 and G_2 , and chooses the start variable q , in a uniform way that makes the preceding properties hold.

This gives us a function **union** \in **Gram** \times **Gram** \rightarrow **Gram**.

We do something similar when defining the other closure operations. In what follows, though, we'll ignore this issue, so as to keep things simple.

Concatenation and Closure Operations

Suppose G_1 and G_2 are grammars. We can define a grammar H such that $L(H) = L(G_1)L(G_2)$ by unioning together the variables and productions of G_1 and G_2 , and adding a new start variable q , along with production

This gives us a function $\text{concat} \in \text{Gram} \times \text{Gram} \rightarrow \text{Gram}$.

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$$q \rightarrow \% \mid s_G q.$$

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Intersection of Grammar and EFA

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For all $p \in Q_G$ and $q, r \in Q_M$, H has a variable $\langle p, q, r \rangle$ that generates

$$\{ w \in (\mathbf{alphabet } G)^* \mid w \in \Pi_{G,p} \text{ and } \quad \}.$$

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For each $\%$ -production $p \rightarrow \%$ of G and $q, r \in Q_M$, if $\langle p, q, r \rangle \in \Delta_M(\{q\}, \%)$, then H will have the production

$$\langle p, q, r \rangle \rightarrow \%.$$

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Intersection of Grammar and EFA

To say what the remaining productions of H are, define a function

$$f \in (\mathbf{alphabet} G \cup Q_G) \times Q_M \times Q_M \rightarrow \mathbf{alphabet} G \cup Q_H$$

by: for all $a \in \mathbf{alphabet} G \cup Q_G$ and $q, r \in Q_M$,

$$f(a, q, r) = \begin{cases} a, & \text{if } a \in \mathbf{alphabet} G, \text{ and} \\ \langle a, q, r \rangle, & \text{if } a \in Q_G. \end{cases}$$

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For all $p \in Q_G$, $n \in \mathbb{N} - \{0\}$, $a_1, \dots, a_n \in \mathbf{Sym}$ and $q_1, \dots, q_{n+1} \in Q_M$, if

- $p \rightarrow a_1 \cdots a_n \in P_G$, and
- for all $i \in [1 : n]$, if $a_i \in \mathbf{alphabet} G$, then $q_{i+1} \in \Delta_M(\{q_i\}, a_i)$,

then we let

$$\langle p, q_1, q_{n+1} \rangle \rightarrow f(a_1, q_1, q_2) \cdots f(a_n, q_n, q_{n+1})$$

be production of H .

Intersection of Grammar and EFA

This gives us a function **inter** \in **Gram** \times **EFA** \rightarrow **Gram**.

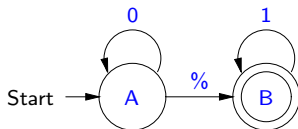
Intersection of Grammar and EFA

This gives us a function $\text{inter} \in \text{Gram} \times \text{EFA} \rightarrow \text{Gram}$.

For example, let G be the grammar

$$A \rightarrow \% \mid 0A1A \mid 1A0A,$$

and M be the EFA



so that G generates all elements of $\{0, 1\}^*$ with an equal number of 0's and 1's, and M accepts $\{0\}^* \{1\}^*$.

Intersection of Grammar and EFA

Then **simplify** H is

$$\begin{aligned}A &\rightarrow \langle A, A, B \rangle, \\ \langle A, A, A \rangle &\rightarrow \%, \\ \langle A, A, B \rangle &\rightarrow \%, \\ \langle A, A, B \rangle &\rightarrow 0\langle A, A, A \rangle 1\langle A, B, B \rangle, \\ \langle A, A, B \rangle &\rightarrow 0\langle A, A, B \rangle 1\langle A, B, B \rangle, \\ \langle A, A, B \rangle &\rightarrow 0\langle A, B, B \rangle 1\langle A, B, B \rangle, \\ \langle A, B, B \rangle &\rightarrow \%.\end{aligned}$$

Note that simplification eliminated the variable $\langle A, B, A \rangle$. If we hand simplify further, we can turn this into:

$$\begin{aligned}A &\rightarrow \langle A, A, B \rangle, \\ \langle A, A, B \rangle &\rightarrow \% \mid 0\langle A, A, B \rangle 1\end{aligned}$$

Intersection of Grammar and EFA

To prove that our intersection algorithm is correct, we'll need two lemmas.

Lemma 4.7.1

For $p \in Q_G$, let the property $P_p(w)$, for $w \in \Pi_{G,p}$, be:

for all $q, r \in Q_M$, if $r \in \Delta_M(\{q\}, w)$, then $w \in \Pi_{H, \langle p, q, r \rangle}$.

Then, for all $p \in Q_G$, for all $w \in \Pi_{G,p}$, $P_p(w)$.

Proof. By induction on Π .

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Then, for all $p \in Q_G$, for all $w \in \Pi_{G,p}$, $P_p(w)$.

Proof. By induction on Π . In (2), we use the fact that, if $n \in \mathbb{N} - \{0\}$, $q_1, q_{n+1} \in Q_M$, $w_1, \dots, w_n \in \mathbf{Str}$ and $q_{n+1} \in \Delta_M(\{q_1\}, w_1 \cdots w_n)$, then there are $q_2, \dots, q_n \in Q_M$ such that $q_{i+1} \in \Delta_M(\{q_i\}, w_i)$, for all $i \in [1 : n]$.

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Intersection of Grammar and EFA

Lemma 4.7.2

Let the property $P_A(w)$, for $w \in \Pi_{H,A}$, be

$$w \in L(G) \text{ and } w \in L(M).$$

For $p \in Q_G$ and $q, r \in Q_M$, let the property $P_{\langle p,q,r \rangle}(w)$, for $w \in \Pi_{H,\langle p,q,r \rangle}$, be

$$w \in \Pi_{G,p} \text{ and } r \in \Delta_M(\{q\}, w).$$

Then:

- (1) For all $w \in \Pi_{H,A}$, $P_A(w)$.
- (2) For all $p \in Q_G$ and $q, r \in Q_M$, for all $w \in \Pi_{H,\langle p,q,r \rangle}$, $P_{\langle p,q,r \rangle}(w)$.

Proof. We proceed by induction on Π . \square

Intersection of Grammar and EFA

Lemma 4.7.3

$$L(H) = L(G) \cap L(M).$$

Proof. $L(H) \subseteq L(G) \cap L(M)$ follows by Lemma 4.7.2(1).



Intersection of Grammar and EFA

Lemma 4.7.3

$$L(H) = L(G) \cap L(M).$$

Proof. $L(H) \subseteq L(G) \cap L(M)$ follows by Lemma 4.7.2(1).

For the other inclusion, suppose $w \in L(G) \cap L(M)$, so that $w \in \Pi_{G, s_M}$ and $r \in \Delta_M(\{s_M\}, w)$, for some $r \in A_M$. By Lemma 4.7.1, it follows that $w \in \Pi_{H, \langle s_G, s_M, r \rangle}$. But because $r \in A_M$, we have that $A \rightarrow \langle s_G, s_M, r \rangle$ is a production of H . Thus $w \in \Pi_{H, A} = L(H)$. \square

Difference of Grammar and DFA

Given a grammar G and a DFA M , we can define the difference of G and M to be

$$\text{inter}(G, \text{complement}(M, \quad)).$$

This is analogous to what we did when defining the difference of DFAs.

This gives us a function $\text{minus} \in \text{Gram} \times \text{DFA} \rightarrow \text{Gram}$.

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This gives us a function **$\text{minus} \in \text{Gram} \times \text{DFA} \rightarrow \text{Gram}$** .

Summary of Closure Properties

Theorem 4.7.1

Suppose $L, L_1, L_2 \in \mathbf{CFLan}$ and $L' \in \mathbf{RegLan}$. Then:

- (1) $L_1 \cup L_2 \in \mathbf{CFLan}$;
- (2) $L_1 L_2 \in \mathbf{CFLan}$;
- (3) $L^* \in \mathbf{CFLan}$;
- (4) $L \cap L' \in \mathbf{CFLan}$; and
- (5) $L - L' \in \mathbf{CFLan}$.

Operations on Grammars in Forlan

The Forlan module `Gram` defines the following constants and operations on grammars:

```
val emptyStr    : gram
val emptySet    : gram
val fromStr     : str -> gram
val fromSym     : sym -> gram
val union      : gram * gram -> gram
val concat     : gram * gram -> gram
val closure    : gram -> gram
val fromStrSet : str set -> gram
val inter      : gram * efa -> gram
val minus      : gram * dfa -> gram
```

The functions `fromStr` and `fromSym` are also available in the top-level environment with the names

```
val strToGram : str -> gram
val symToGram : sym -> gram
```

Forlan Examples

For example, we can construct a grammar G such that $L(G) = \{01\} \cup \{10\}\{11\}^*$, as follows.

```
- val gram1 = strToGram(Str.fromString "01");  
val gram1 = - : gram  
- val gram2 = strToGram(Str.fromString "10");  
val gram2 = - : gram  
- val gram3 = strToGram(Str.fromString "11");  
val gram3 = - : gram  
- val gram =  
= Gram.union(gram1,  
=           Gram.concat(gram2,  
=                       Gram.closure gram3));  
val gram = - : gram
```

Forlan Examples

```
- val gram' = Gram.renameVariablesCanonically gram;  
val gram' = - : gram  
- Gram.output("", gram');  
{variables} A, B, C, D, E, F {start variable} A  
{productions}  
A -> B | C; B -> 01; C -> DE; D -> 10; E -> % | FE;  
F -> 11  
val it = () : unit
```


Forlan Examples

We can use `Gram.fromStrSet` as follows:

```
- val gram'' =  
= Gram.fromStrSet  
= (StrSet.fromString "0, 01, 010, 0101");  
val gram'' = - : gram  
- val gram''' = Gram.renameVariablesCanonically  
gram'';  
val gram''' = - : gram  
- Gram.output("", gram''');  
{variables} A, B, C, D, E {start variable} A  
{productions}  
A -> B | C | D | E; B -> 0; C -> 01; D -> 010;  
E -> 0101  
val it = () : unit
```

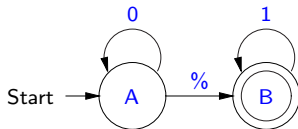
Forlan Examples

And here are examples of how we can use `Gram.inter` and `Gram.minus`.

Let `gram` be the grammar

$$A \rightarrow \% \mid 0A1A \mid 1A0A,$$

and `efa` be the EFA



Forlan Examples

```
- val gram' = Gram.inter(gram, efa);
val gram' = - : gram
- Gram.output("", gram');
{variables} A, <A,A,A>, <A,A,B>, <A,B,B>
{start variable} A
{productions}
A -> <A,A,B>; <A,A,A> -> %;
<A,A,B> ->
% | 0<A,A,A>1<A,B,B> | 0<A,A,B>1<A,B,B> |
0<A,B,B>1<A,B,B>;
<A,B,B> -> %
val it = () : unit
```

Forlan Examples

```
- val gram'' =  
=   Gram.eliminateVariable  
=   (Gram.eliminateVariable  
=     (gram', Sym.fromString "<A,A,A>"),  
=     Sym.fromString "<A,B,B>");  
val gram'' = - : gram  
- val gram''' =  
=   Gram.renameVariablesCanonically  
=   (Gram.restart(Gram.simplify gram''));  
val gram''' = - : gram  
- Gram.output("", gram''');  
{variables} A {start variable} A  
{productions} A -> % | OA1  
val it = () : unit
```

Forlan Examples

```
- val dfa =  
=       DFA.renameStatesCanonically  
=       (DFA.minimize(nfaToDFA(efaToNFA efa)));  
val dfa = - : dfa  
- val gram'' = Gram.minus(gram, dfa);  
val gram'' = - : gram  
- Gram.generated gram'' (Str.fromString "0101");  
val it = true : bool  
- Gram.generated gram'' (Str.fromString "0011");  
val it = false : bool
```