

## *5.2: Closure Properties of Recursive and Recursively Enumerable Languages*

In this section, we will see that the recursive and recursively enumerable languages are closed under union, concatenation, closure and intersection.

The recursive languages are also closed under set difference and complementation.

In the next section, we will see that the recursively enumerable languages are not closed under complementation or set difference.

On the other hand, we will see in this section that, if a language and its complement are both r.e., then the language is recursive.

## Closure Properties of Recursive Languages

### Theorem 5.2.1

If  $L$ ,  $L_1$  and  $L_2$  are recursive languages, then so are  $L_1 \cup L_2$ ,  $L_1 L_2$ ,  $L^*$ ,  $L_1 \cap L_2$  and  $L_1 - L_2$ .

**Proof.** Let's consider the concatenation case as an example. Since  $L_1$  and  $L_2$  are recursive languages, there are string predicate programs  $pr_1$  and  $pr_2$  that test whether strings are in  $L_1$  and  $L_2$ , respectively.

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**Proof (cont.).** We write a program  $pr$  with form

```
lam(w,  
    letSimp(f1,  
            pr1,  
            letSimp(f2,  
                    pr2,  
                    ... )))
```

which tests whether its input is an element of  $L_1L_2$ .

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**Proof (cont.).** The elided part of *pr* generates all of the pairs of strings  $(x_1, x_2)$  such that  $x_1x_2$  is equal to the value of  $w$ . Then it works through these pairs, one by one.

- Given such a pair  $(x_1, x_2)$ , *pr* calls *f1* with  $x_1$  to check whether  $x_1 \in L_1$ . If the answer is **const(false)**, then it goes on to the next pair.
- Otherwise, it calls *f2* with  $x_2$  to check whether  $x_2 \in L_2$ . If the answer is **const(false)**, then it goes on to the next pair. Otherwise, it returns **const(true)**.
- If *pr* runs out of pairs to check, then it returns **const(false)**.

We can check that *pr* is a string predicate program testing whether  $w \in L_1L_2$ . Thus  $L_1L_2$  is recursive.  $\square$

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### **Corollary 5.2.2**

*If  $\Sigma$  is an alphabet and  $L \subseteq \Sigma^*$  is recursive, then so is  $\Sigma^* - L$ .*

**Proof.** Follows from Theorem 5.2.1, since  $\Sigma^*$  is recursive.  $\square$

# Closure Properties of Recursively Enumerable Languages

## Theorem 5.2.2

If  $L$ ,  $L_1$  and  $L_2$  are recursively enumerable languages, then so are  $L_1 \cup L_2$ ,  $L_1 L_2$ ,  $L^*$  and  $L_1 \cap L_2$ .

**Proof.** We consider the concatenation case as an example. Since  $L_1$  and  $L_2$  are recursively enumerable, there are programs  $pr_1$  and  $pr_2$  such that, for all  $w \in \mathbf{Str}$ ,  $w \in L_1$  iff  $\mathbf{eval}(\mathbf{app}(pr_1, \mathbf{str}(w))) = \mathbf{norm}(\mathbf{const}(\mathbf{true}))$ , and for all  $w \in \mathbf{Str}$ ,  $w \in L_2$  iff  $\mathbf{eval}(\mathbf{app}(pr_2, \mathbf{str}(w))) = \mathbf{norm}(\mathbf{const}(\mathbf{true}))$ . (Remember that  $pr_1$  and  $pr_2$  may fail to terminate on some inputs.)

To show that  $L_1 L_2$  is recursively enumerable, we will construct a program  $pr$  such that, for all  $w \in \mathbf{Str}$ ,  $w \in L_1 L_2$  iff  $\mathbf{eval}(\mathbf{app}(pr, \mathbf{str}(w))) = \mathbf{norm}(\mathbf{const}(\mathbf{true}))$ .

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**Proof (cont.).** When  $pr$  is called with  $\mathbf{str}(w)$ , for some  $w \in \mathbf{Str}$ , it behaves as follows. First, it generates all the pairs of strings  $(x_1, x_2)$  such that  $w = x_1x_2$ . Let these pairs be  $(x_{1,1}, x_{2,1}), \dots, (x_{1,n}, x_{2,n})$ . Now,  $pr$  uses our *incremental* interpretation function to run a fixed number of steps of  $\mathbf{app}(pr_1, \mathbf{str}(x_{1,i}))$  and  $\mathbf{app}(pr_2, \mathbf{str}(x_{2,i}))$  (working with  $\mathbf{app}(pr_1, \mathbf{str}(x_{1,i}))$  and  $\mathbf{app}(pr_2, \mathbf{str}(x_{2,i}))$ ), for all  $i \in [1 : n]$ , and then repeat this over and over again.

- If, at some stage, the incremental interpretation of  $\mathbf{app}(pr_1, \mathbf{str}(x_{1,i}))$  returns  $\mathbf{const(true)}$ , then  $x_{1,i}$  is marked as being in  $L_1$ .
- If, at some stage, the incremental interpretation of  $\mathbf{app}(pr_2, \mathbf{str}(x_{2,i}))$  returns  $\mathbf{const(true)}$ , then the  $x_{2,i}$  is marked as being in  $L_2$ .

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### Proof (cont.).

- If, at some stage, the incremental interpretation of  $\text{app}(pr_1, \text{str}(x_{1,i}))$  returns something other than  $\overline{\text{const}(\text{true})}$ , then the  $i$ 'th pair is marked as discarded.
- If, at some stage, the incremental interpretation of  $\text{app}(pr_2, \text{str}(x_{2,i}))$  returns something other than  $\overline{\text{const}(\text{true})}$ , then the  $i$ 'th pair is marked as discarded.
- If, at some stage,  $x_{1,i}$  is marked as in  $L_1$  and  $x_{2,i}$  is marked as in  $L_2$ , then  $Q$  returns  $\text{const}(\text{true})$ .
- If, at some stage, there are no remaining pairs, then  $pr$  returns  $\text{const}(\text{false})$ .

□

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### Theorem 5.2.3

If  $\Sigma$  is an alphabet,  $L \subseteq \Sigma^*$  is a recursively enumerable language, and  $\Sigma^* - L$  is recursively enumerable, then  $L$  is recursive.

**Proof.** Since  $L$  and  $\Sigma^* - L$  are recursively enumerable languages, there are programs  $pr_1$  and  $pr_2$  such that, for all  $w \in \mathbf{Str}$ ,  $w \in L$  iff  $\mathbf{eval}(\mathbf{app}(pr_1, \mathbf{str}(w))) = \mathbf{norm}(\mathbf{const}(\mathbf{true}))$ , and for all  $w \in \mathbf{Str}$ ,  $w \in \Sigma^* - L$  iff  $\mathbf{eval}(\mathbf{app}(pr_2, \mathbf{str}(w))) = \mathbf{norm}(\mathbf{const}(\mathbf{true}))$ .

We construct a string predicate program  $pr$  that tests whether its input is in  $L$ . Given  $\mathbf{str}(w)$ , for  $w \in \mathbf{Str}$ ,  $pr$  proceeds as follows. If  $w \notin \Sigma^*$ , then  $pr$  returns  $\mathbf{const}(\mathbf{false})$ . Otherwise,  $pr$  alternates between incrementally interpreting  $\mathbf{app}(pr_1, \mathbf{str}(w))$  (working with  $\underline{\mathbf{app}(pr_1, \mathbf{str}(w))}$ ) and  $\mathbf{app}(pr_2, \mathbf{str}(w))$  (working with  $\mathbf{app}(pr_2, \mathbf{str}(w))$ ).

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### **Proof (cont.).**

- If, at some stage, the first incremental interpretation returns const(true), then *pr* returns **const(true)**.
- If, at some stage, the second incremental interpretation returns const(true), then *pr* returns **const(false)**.
- If, at some stage, the first incremental interpretation returns anything other than const(true), then *pr* returns **const(false)**.
- If, at some stage, the second incremental interpretation returns anything other than const(true), then *pr* returns **const(true)**.

□