CIS 705 — Programming Languages — Spring 2010

Assignment 1

Model Answers

Exercise 1

As usual, we say that a term t is a *normal form* iff there is no term t' such that $t \to t'$.

We begin by proving two lemmas.

Lemma 1.1
For all numeric values nv, nv is a normal form.

Proof. Define a predicate P on numeric values by: $P(nv)$ iff nv is a normal form. We use the principle of structural induction on numeric values to prove that, for all numeric values nv, $P(nv)$.

(Zero) We must show $P(\mathsf{0})$, i.e., that $\mathsf{0}$ is a normal form. This follows by the inversion lemma for the evaluation relation, because none of the evaluation rules apply to $\mathsf{0}$.

(Successor) Suppose nv is a numeric value, and assume the inductive hypothesis, $P(nv)$, i.e., nv is a normal form. We must show $P(\mathsf{succ}\ nv)$, i.e., $\mathsf{succ}\ nv$ is a normal form. Suppose, toward a contradiction, that $\mathsf{succ}\ nv$ is not a normal form. Thus $\mathsf{succ}\ nv \to t$ for some term t. Applying the inversion lemma for the evaluation relation, only case (Succ) could apply, so that $t = \mathsf{succ}\ t'$, for some t' such that $nv \to t'$. But this contradicts the fact that nv is a normal form. Thus we have that $\mathsf{succ}\ nv$ is a normal form.

□

Lemma 1.2
For all answers a, a is a normal form.

Proof. If a is true, false or error, the inversion lemma for the evaluation relation shows that a is a normal form. Otherwise, a is a numeric value, and the result follows by Lemma 1.1. □

Now we use our lemmas to prove the exercise's result. Define a predicate P on pairs of terms by: $P(t_1, t_2)$ iff, for all terms t_2', if $t_1 \to t_2'$, then $t_2 = t_2'$. It will suffice to show that, for all terms t_1 and t_2, if $t_1 \to t_2$, then $P(t_1, t_2)$. (To see that this is so, suppose t_1, t_2 and t_2' are terms, $t_1 \to t_2$ and $t_1 \to t_2'$. Then $P(t_1, t_2)$, so that $t_2 = t_2'$.) We proceed using the principle of induction on the evaluation relation.

(IfTrue) Suppose t_2 and t_3 are terms. We must show that $P(\mathsf{if\ true\ then}\ t_2\ \mathsf{else}\ t_3, t_2)$. Suppose t is a term and $\mathsf{if\ true\ then}\ t_2\ \mathsf{else}\ t_3 \to t$. We must show that $t_2 = t$. By Lemma 1.2, we have that true is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (IfTrue) can apply, so that $t = t_2$, i.e., $t_2 = t$.

(IfFalse) Similar to (IfTrue).

(IfNum) Suppose nv is a numeric value and t_2 and t_3 are terms. We must show that $P(\mathsf{if}\ nv\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3, \mathsf{error})$. Suppose t is a term and $\mathsf{if}\ nv\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to t$. We must show that $\mathsf{error} = t$. By Lemma 1.2, we have that nv is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (IfNum) can apply, so that $t = \mathsf{error}$, i.e., $\mathsf{error} = t$.

(IfError) Similar to (IfNum).

(If) Suppose t_1, t_1', t_2 and t_3 are terms, $t_1 \to t_1'$, and assume the inductive hypothesis, $P(t_1, t_1')$. We must show that $P(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3,\ \mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3)$. Suppose t is a term and $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to t$. We must show that $\mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 = t$. By Lemma 1.2, because t_1 is not a normal form, we have that t_1 is not an answer. Thus, applying the inversion lemma for the evaluation relation, only case (If) can apply, so that $t = \mathsf{if}\ t_1''\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$, for some term t_1'' such that $t_1 \to t_1''$. By the inductive hypothesis, it follows that $t_1' = t_1''$. Thus $\mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 = \mathsf{if}\ t_1''\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 = t$.

(SuccBool) Suppose bv is a boolean value. We must show that $P(\mathsf{succ}\ bv, \mathsf{error})$. Suppose t is a term and $\mathsf{succ}\ bv \to t$. We must show that $\mathsf{error} = t$. By Lemma 1.2, we have that bv is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (SuccBool) can apply, so that $t = \mathsf{error}$, i.e., $\mathsf{error} = t$.

(SuccError) We must show that $P(\mathsf{succ}\ \mathsf{error}, \mathsf{error})$. Suppose t is a term and $\mathsf{succ}\ \mathsf{error} \to t$. We must show that $\mathsf{error} = t$. By Lemma 1.2, we have that error is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (SuccError) can apply, so that $t = \mathsf{error}$, i.e., $\mathsf{error} = t$.

(Succ) Suppose t_1 and t_1' are terms, $t_1 \to t_1'$ and assume the inductive hypothesis, $P(t_1, t_1')$. We must show that $P(\mathsf{succ}\ t_1, \mathsf{succ}\ t_1')$. Suppose t is a term and $\mathsf{succ}\ t_1 \to t$. We must show that $\mathsf{succ}\ t_1' = t$. By Lemma 1.2, because t_1 is not a normal form, we have that t_1 is not an answer. Thus, applying the inversion lemma for the evaluation relation, only case (Succ) can apply, so that $t = \mathsf{succ}\ t_1''$, for some term t_1'' such that $t_1 \to t_1''$. By the inductive hypothesis, it follows that $t_1' = t_1''$. Thus $\mathsf{succ}\ t_1' = \mathsf{succ}\ t_1'' = t$.

(PredBool) Suppose bv is a boolean value. We must show that $P(\mathsf{pred}\ bv, \mathsf{error})$. Suppose t is a term and $\mathsf{pred}\ bv \to t$. We must show that $\mathsf{error} = t$. By Lemma 1.2, we have that bv is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (PredBool) can apply, so that $t = \mathsf{error}$, i.e., $\mathsf{error} = t$.

(PredZero) We must show that $P(\mathsf{pred}\ 0, \mathsf{error})$. Suppose t is a term and $\mathsf{pred}\ 0 \to t$. We must show that $\mathsf{error} = t$. By Lemma 1.2, we have that 0 is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (PredZero) can apply, so that $t = \mathsf{error}$, i.e., $\mathsf{error} = t$.

(PredSucc) Suppose nv is a numeric value. We must show that $P(\mathsf{pred}(\mathsf{succ}\ nv), nv)$. Suppose t is a term and $\mathsf{pred}(\mathsf{succ}\ nv) \to t$. We must show that $nv = t$. By Lemma 1.2, we have that $\mathsf{succ}\ nv$ is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (PredSucc) can apply, so that $t = nv$, i.e., $nv = t$.

(PredError) We must show that $P(\mathsf{pred\ error}, \mathsf{error})$. Suppose t is a term and $\mathsf{pred\ error} \to t$. We must show that $\mathsf{error} = t$. By Lemma 1.2, we have that error is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (PredError) can apply, so that $t = \mathsf{error}$, i.e., $\mathsf{error} = t$.

(Pred) Suppose t_1 and t_1' are terms, $t_1 \to t_1'$ and assume the inductive hypothesis, $P(t_1, t_1')$. We must show that $P(\mathsf{pred}\ t_1, \mathsf{pred}\ t_1')$. Suppose t is a term and $\mathsf{pred}\ t_1 \to t$. We must show that $\mathsf{pred}\ t_1' = t$. By Lemma 1.2, because t_1 is not a normal form, we have that t_1 is not an answer. Thus, applying the inversion lemma for the evaluation relation, only case (Pred) can apply, so that $t = \mathsf{pred}\ t_1''$, for some term t_1'' such that $t_1 \to t_1''$. By the inductive hypothesis, it follows that $t_1' = t_1''$. Thus $\mathsf{pred}\ t_1' = \mathsf{pred}\ t_1'' = t$.

(IszeroBool) Similar to (PredBool).

(IszeroZero) We must show that $P(\mathsf{iszero}\ \mathsf{0}, \mathsf{true})$. Suppose t is a term and $\mathsf{iszero}\ \mathsf{0} \to t$. We must show that $\mathsf{true} = t$. By Lemma 1.2, we have that $\mathsf{0}$ is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (IszeroZero) can apply, so that $t = \mathsf{true}$, i.e., $\mathsf{true} = t$.

(IszeroSucc) Suppose nv is a numeric value. We must show that $P(\mathsf{iszero}(\mathsf{succ}\ nv), \mathsf{false})$. Suppose t is a term and $\mathsf{iszero}(\mathsf{succ}\ nv) \to t$. We must show that $\mathsf{false} = t$. By Lemma 1.2, we have that $\mathsf{succ}\ nv$ is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (IszeroSucc) can apply, so that $t = \mathsf{false}$, i.e., $\mathsf{false} = t$.

(IszeroError) Similar to (PredError).

(Iszero) Similar to (Pred).

(OtherwiseValue) Suppose t is a term and v is a value. We must show that $P(v\ \mathsf{otherwise}\ t, v)$. Suppose t' is a term and $v\ \mathsf{otherwise}\ t \to t'$. We must show that $v = t'$. By Lemma 1.2, we have that v is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (OtherwiseValue) can apply, so that $t' = v$, i.e., $v = t'$.

(OtherwiseError) Suppose t is a term. We must show that $P(\mathsf{error\ otherwise}\ t, t)$. Suppose t' is a term and $\mathsf{error\ otherwise}\ t \to t'$. We must show that $t = t'$. By Lemma 1.2, we have that error is a normal form. Thus, applying the inversion lemma for the evaluation relation, only case (OtherwiseError) can apply, so that $t' = t$, i.e., $t = t'$.

(Otherwise) Suppose t_1, t_2 and t_1' are terms, $t_1 \to t_1'$, and assume the inductive hypothesis, $P(t_1, t_1')$. We must show that $P(t_1\ \mathsf{otherwise}\ t_2,\ t_1'\ \mathsf{otherwise}\ t_2)$. Suppose t is a term and $t_1\ \mathsf{otherwise}\ t_2 \to t$. We must show that $t_1'\ \mathsf{otherwise}\ t_2 = t$. By Lemma 1.2, because t_1 is not a normal form, we have that t_1 is not an answer. Thus, applying the inversion lemma for the evaluation relation, only case (Otherwise) can apply, so that $t = t_1''\ \mathsf{otherwise}\ t_2$, for some term t_1'' such that $t_1 \to t_1''$. By the inductive hypothesis, it follows that $t_1' = t_1''$. Thus $t_1'\ \mathsf{otherwise}\ t_2 = t_1''\ \mathsf{otherwise}\ t_2 = t$.

Exercise 2

Define a predicate P on pairs of terms and answers by: $P(t, a)$ iff, for all answers a', if $t \Rightarrow a'$, then $a = a'$. It will suffice show that, for all terms t and answers a, if $t \Rightarrow a$, then $P(t, a)$. (To see that this is so, suppose t is a term, a and a' are answers, $t \Rightarrow a$ and $t \Rightarrow a'$. Then $P(t, a)$, so that $a = a'$.) We proceed using the principle of induction on the complete evaluation relation.

(True) We must show that $P(\mathsf{true}, \mathsf{true})$. Suppose a is an answer and $\mathsf{true} \Rightarrow a$. We must show that $\mathsf{true} = a$. Applying the inversion lemma for the complete evaluation relation, only case (True) can apply, so that $a = \mathsf{true}$, i.e., $\mathsf{true} = a$.

(False) Similar to (True).

(Zero) Similar to (True).

(Error) Similar to (True).

(IfTrue) Suppose t_1, t_2 and t_3 are terms, a is an answer, $t_1 \Rightarrow \mathsf{true}$ and $t_2 \Rightarrow a$, and assume the inductive hypothesis, $P(t_1, \mathsf{true})$ and $P(t_2, a)$. We must show that $P(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3, a)$. Suppose a' is an answer and $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \Rightarrow a'$. We must show that $a = a'$. Because $P(t_1, \mathsf{true})$, it follows that t_1 doesn't completely evaluate to false, a numeric value or error. Thus, applying the inversion lemma for the complete evaluation relation, only case (IfTrue) applies, so that there is an answer a'' such that $t_2 \Rightarrow a''$ and $a' = a''$. But $P(t_2, a)$, and thus $a = a'' = a'$.

(IfFalse) Similar to (IfTrue).

(IfNum) Suppose t_1, t_2 and t_3 are terms, nv is a numeric value and $t_1 \Rightarrow nv$, and assume the inductive hypothesis, $P(t_1, nv)$. We must show that $P(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3, \mathsf{error})$. Suppose a is an answer and $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \Rightarrow a$. We must show that $\mathsf{error} = a$. Because $P(t_1, nv)$, it follows that t_1 doesn't completely evaluate to a boolean value or error. Thus, applying the inversion lemma for the complete evaluation relation, only case (IfNum) applies, so that $a = \mathsf{error}$, i.e., $\mathsf{error} = a$.

(IfError) Suppose t_1, t_2 and t_3 are terms and $t_1 \Rightarrow \mathsf{error}$, and assume the inductive hypothesis, $P(t_1, \mathsf{error})$. We must show that $P(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3, \mathsf{error})$. Suppose a is an answer and $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \Rightarrow a$. We must show that $\mathsf{error} = a$. Because $P(t_1, \mathsf{error})$, it follows that t_1 doesn't completely evaluate to a numeric or boolean value. Thus, applying the inversion lemma for the complete evaluation relation, only case (IfError) applies, so that $a = \mathsf{error}$, i.e., $\mathsf{error} = a$.

(SuccBool) Suppose t is a term, bv is a boolean value and $t \Rightarrow bv$, and assume the inductive hypothesis, $P(t, bv)$. We must show that $P(\mathsf{succ}\ t, \mathsf{error})$. Suppose a is an answer and $\mathsf{succ}\ t \Rightarrow a$. We must show that $\mathsf{error} = a$. Because $P(t, bv)$, it follows that t doesn't completely evaluate to a numeric value or error. Thus, applying the inversion lemma for the complete evaluation relation, only case (SuccBool) applies, so that $a = \mathsf{error}$, i.e., $\mathsf{error} = a$.

(SuccNum) Suppose t is a term, nv is a numeric value and $t \Rightarrow nv$, and assume the inductive hypothesis, $P(t, nv)$. We must show that $P(\mathsf{succ}\ t, \mathsf{succ}\ nv)$. Suppose a is an answer and $\mathsf{succ}\ t \Rightarrow a$. We must show that $\mathsf{succ}\ nv = a$. Because $P(t, nv)$, it follows that t doesn't completely evaluate to a boolean or error. Thus, applying the inversion lemma for the complete evaluation relation, only case (SuccNum) applies, so there is a numeric value nv' such that $t \Rightarrow nv'$ and $a = \mathsf{succ}\ nv'$. But $P(t, nv)$, and thus $nv = nv'$. Thus $\mathsf{succ}\ nv = \mathsf{succ}\ nv' = a$.

(SuccError) Suppose t is a term and $t \Rightarrow \mathsf{error}$, and assume the inductive hypothesis, $P(t, \mathsf{error})$. We must show that $P(\mathsf{succ}\ t, \mathsf{error})$. Suppose a is an answer and $\mathsf{succ}\ t \Rightarrow a$. We must show that $\mathsf{error} = a$. Because $P(t, \mathsf{error})$, it follows that t doesn't completely evaluate to a boolean or numeric value. Thus, applying the inversion lemma for the complete evaluation relation, only case (SuccError) applies, so that $a = \mathsf{error}$, i.e., $\mathsf{error} = a$.

(PredBool) Similar to (SuccBool).

(PredZero) Suppose t is a term and $t \Rightarrow \mathsf{0}$, and assume the inductive hypothesis, $P(t, \mathsf{0})$. We must show that $P(\mathsf{pred}\ t, \mathsf{error})$. Suppose a is an answer and $\mathsf{pred}\ t \Rightarrow a$. We must show that $\mathsf{error} = a$. Because $P(t, \mathsf{0})$, it follows that t doesn't completely evaluate to a boolean value, a numeric value other than $\mathsf{0}$, or error. Thus, applying the inversion lemma for the complete evaluation relation, only case (PredZero) applies, so that $a = \mathsf{error}$, i.e., $\mathsf{error} = a$.

(PredSucc) Suppose t is a term, nv is a numeric value and $t \Rightarrow \mathsf{succ}\ nv$, and assume the inductive hypothesis, $P(t, \mathsf{succ}\ nv)$. We must show that $P(\mathsf{pred}\ t, nv)$. Suppose a is an answer and $\mathsf{pred}\ t \Rightarrow a$. We must show that $nv = a$. Because $P(t, \mathsf{succ}\ nv)$, it follows that t doesn't completely evaluate to a boolean value, $\mathsf{0}$ or error. Thus, applying the inversion lemma for the complete evaluation relation, only case (PredSucc) applies, so that there is a numeric value nv' such that $t \Rightarrow \mathsf{succ}\ nv'$ and $a = nv'$. But $P(t, \mathsf{succ}\ nv)$, and thus $\mathsf{succ}\ nv = \mathsf{succ}\ nv'$, so that $nv = nv' = a$.

(PredError) Similar to (SuccError).

(IszeroBool) Similar to (SuccBool).

(IszeroZero) Suppose t is a term and $t \Rightarrow \mathsf{0}$, and assume the inductive hypothesis, $P(t, \mathsf{0})$. We must show that $P(\mathsf{iszero}\ t, \mathsf{true})$. Suppose a is an answer and $\mathsf{iszero}\ t \Rightarrow a$. We must show that $\mathsf{true} = a$. Because $P(t, \mathsf{0})$, it follows that t doesn't completely evaluate to a boolean value, a numeric value other than $\mathsf{0}$, or error. Thus, applying the inversion lemma for the complete evaluation relation, only case (IszeroZero) applies, so that $a = \mathsf{true}$, i.e., $\mathsf{true} = a$.

(IszeroSucc) Suppose t is a term, nv is a numeric value and $t \Rightarrow \mathsf{succ}\ nv$, and assume the inductive hypothesis, $P(t, \mathsf{succ}\ nv)$. We must show that $P(\mathsf{iszero}\ t, \mathsf{false})$. Suppose a is an answer and $\mathsf{iszero}\ t \Rightarrow a$. We must show that $\mathsf{false} = a$. Because $P(t, \mathsf{succ}\ nv)$, it follows that t doesn't completely evaluate to a boolean value, $\mathsf{0}$ or error. Thus, applying the inversion lemma for the complete evaluation relation, only case (PredSucc) applies, so that $a = \mathsf{false}$, i.e., $\mathsf{false} = a$.

(IszeroError) Similar to (SuccError).

(OtherwiseValue) Suppose t_1 and t_2 are terms, v is a value and $t_1 \Rightarrow v$, and assume the inductive hypothesis, $P(t_1, v)$. We must show that $P(t_1 \ \mathsf{otherwise}\ t_2, v)$. Suppose a is a value and $t_1 \ \mathsf{otherwise}\ t_2 \Rightarrow a$. We must show that $v = a$. Because $P(t_1, v)$, it follows that t_1 doesn't completely evaluate to error. Thus, applying the inversion lemma for the complete evaluation relation, only case (OtherwiseValue) applies, so that there is a value v' such that $t_1 \Rightarrow v'$ and $a = v'$. But $P(t_1, v)$, and thus $v = v' = a$.

(OtherwiseError) Suppose t_1 and t_2 are terms, a is an answer, $t_1 \Rightarrow \mathsf{error}$ and $t_2 \Rightarrow a$, and assume the inductive hypothesis, $P(t_1, \mathsf{error})$ and $P(t_2, a)$. We must show that $P(t_1 \ \mathsf{otherwise}\ t_2, a)$. Suppose a' is an answer and $t_1 \ \mathsf{otherwise}\ t_2 \Rightarrow a'$. We must show that $a = a'$. Because $P(t_1, \mathsf{error})$, it follows that t_1 does not completely evaluate to a value. Thus, applying the inversion lemma for the complete evaluation relation, only case (OtherwiseError) applies, so that there is an answer a'' such that $t_2 \Rightarrow a''$ and $a' = a''$. But $P(t_2, a)$, and thus $a = a'' = a'$.

Exercise 3

Lemma 3.1
For all answers a, $a \Rightarrow a$.

Proof. By rules (CE-True), (CE-False) and (CE-Error), we have that $\mathsf{true} \Rightarrow \mathsf{true}$, $\mathsf{false} \Rightarrow \mathsf{false}$ and $\mathsf{error} \Rightarrow \mathsf{error}$. Thus it remains to show that, for all numeric values nv, $nv \Rightarrow nv$. Define a predicate P on numeric values by: $P(nv)$ iff $nv \Rightarrow nv$. It will suffice to show that, for all numeric values nv, $P(nv)$. We proceed using the principle of structural induction on numeric values.

(Zero) We must show $P(\mathsf{0})$, i.e., $\mathsf{0} \Rightarrow \mathsf{0}$, and this follows by rule (CE-Zero).

(Successor) Suppose nv is a numeric value, and assume the inductive hypothesis, $P(nv)$. We must show that $P(\mathsf{succ}\ nv)$. Because $P(nv)$, we have that $nv \Rightarrow nv$, so that $\mathsf{succ}\ nv \Rightarrow \mathsf{succ}\ nv$, by rule (CE-SuccNum). Thus $P(\mathsf{succ}\ nv)$.

□

Lemma 3.2
For all terms t_1 and t_2 and answers a, if $t_1 \rightarrow t_2 \Rightarrow a$, then $t_1 \Rightarrow a$.

Proof. Define a predicate P on pairs of terms by: $P(t_1, t_2)$ iff, for all answers a, if $t_2 \Rightarrow a$, then $t_1 \Rightarrow a$. It will suffice to show that, for all terms t_1 and t_2, if $t_1 \rightarrow t_2$, then $P(t_1, t_2)$. (To see that this is so, suppose t_1 and t_2 are terms, a is an answer and $t_1 \rightarrow t_2 \Rightarrow a$. Then $P(t_1, t_2)$. But $t_2 \Rightarrow a$, and thus $t_1 \Rightarrow a$.) We proceed using the principle of induction on the evaluation relation.

(IfTrue) Suppose t_2 and t_3 are terms. We must show that $P(\mathsf{if\ true\ then}\ t_2 \ \mathsf{else}\ t_3, t_2)$. Suppose a is an answer, and $t_2 \Rightarrow a$. We must show that $\mathsf{if\ true\ then}\ t_2 \ \mathsf{else}\ t_3 \Rightarrow a$. By rule (CE-True), we have that $\mathsf{true} \Rightarrow \mathsf{true}$. Then, since $\mathsf{true} \Rightarrow \mathsf{true}$ and $t_2 \Rightarrow a$, rule (CE-IfTrue) shows us that $\mathsf{if\ true\ then}\ t_2 \ \mathsf{else}\ t_3 \Rightarrow a$.

(IfFalse) Similar to (IfTrue).

(IfNum) Suppose nv is a numeric value and t_2 and t_3 are terms. We must show that $P(\mathsf{if}\ nv\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3, \mathsf{error})$. Suppose a is an answer, and $\mathsf{error} \Rightarrow a$. We must show that $\mathsf{if}\ nv\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \Rightarrow a$. By rule (CE-Error), we have that $\mathsf{error} \Rightarrow \mathsf{error}$. Since $\mathsf{error} \Rightarrow \mathsf{error}$ and $\mathsf{error} \Rightarrow a$, Exercise 2 tells us that $\mathsf{error} = a$. By Lemma 3.1, we have that $nv \Rightarrow nv$. Thus rule (CE-IfNum) shows us that $\mathsf{if}\ nv\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \Rightarrow \mathsf{error} = a$.

(IfError) Similar to (IfNum).

(If) Suppose t_1, t_1', t_2 and t_3 are terms, $t_1 \rightarrow t_1'$, and assume the inductive hypothesis, $P(t_1, t_1')$. We must show that $P(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3,\ \mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3)$. Suppose a is an answer, and $\mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \Rightarrow a$. We must show that $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \Rightarrow a$. Applying the inversion lemma for the complete evaluation lemma to $\mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \Rightarrow a$, there are four cases to consider.

- **(IfTrue)** Suppose $t_1' \Rightarrow \mathsf{true}$ and $t_2 \Rightarrow a$. Since $P(t_1, t_1')$, we have that $t_1 \Rightarrow \mathsf{true}$. Thus, because $t_1 \Rightarrow \mathsf{true}$ and $t_2 \Rightarrow a$, rule (CE-IfTrue) shows us that $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \Rightarrow a$.
- **(IfFalse)** Similar to (IfTrue).
- **(IfNum)** Suppose there is a numeric value nv such that $t_1' \Rightarrow nv$ and $a = \mathsf{error}$. Since $P(t_1, t_1')$, we have that $t_1 \Rightarrow nv$. Thus rule (CE-IfNum) shows us that $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \Rightarrow \mathsf{error} = a$.
- **(IfError)** Suppose $t_1' \Rightarrow \mathsf{error}$ and $a = \mathsf{error}$. Since $P(t_1, t_1')$, we have that $t_1 \Rightarrow \mathsf{error}$. Thus rule (CE-IfError) shows us that $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \Rightarrow \mathsf{error} = a$.

(SuccBool) Suppose bv is a boolean value. We must show that $P(\mathsf{succ}\ bv, \mathsf{error})$. Suppose a is an answer, and $\mathsf{error} \Rightarrow a$. We must show that $\mathsf{succ}\ bv \Rightarrow a$. By rule (CE-Error) and Exercise 2, we have that $a = \mathsf{error}$. By Lemma 3.1, we have that $bv \Rightarrow bv$. Thus, by rule (CE-SuccBool), we have that $\mathsf{succ}\ bv \Rightarrow \mathsf{error} = a$.

(SuccError) We must show that $P(\mathsf{succ}\ \mathsf{error}, \mathsf{error})$. Suppose a is an answer, and $\mathsf{error} \Rightarrow a$. We must show that $\mathsf{succ}\ \mathsf{error} \Rightarrow a$. By rule (CE-Error) and Exercise 2, we have that $a = \mathsf{error}$. By rule (CE-Error), we have that $\mathsf{error} \Rightarrow \mathsf{error}$. Thus, by rule (CE-SuccError), we have that $\mathsf{succ}\ \mathsf{error} \Rightarrow \mathsf{error} = a$.

(Succ) Suppose t_1 and t_1' are terms, $t_1 \rightarrow t_1'$ and assume the inductive hypothesis, $P(t_1, t_1')$. We must show that $P(\mathsf{succ}\ t_1, \mathsf{succ}\ t_1')$. Suppose a is an answer, and $\mathsf{succ}\ t_1' \Rightarrow a$. We must show that $\mathsf{succ}\ t_1 \Rightarrow a$. Applying the inversion lemma for the complete evaluation relation to $\mathsf{succ}\ t_1' \Rightarrow a$, there are three cases to consider.

- **(SuccBool)** Suppose there is a boolean value bv such that $t_1' \Rightarrow bv$ and $a = \mathsf{error}$. Since $P(t_1, t_1')$, it follows that $t_1 \Rightarrow bv$. Thus rule (CE-SuccBool) shows us that $\mathsf{succ}\ t_1 \Rightarrow \mathsf{error} = a$.
- **(SuccNum)** Suppose there is a numeric value nv such that $t_1' \Rightarrow nv$ and $a = \mathsf{succ}\ nv$. Since $P(t_1, t_1')$, it follows that $t_1 \Rightarrow nv$. Thus rule (CE-SuccNum) shows us that $\mathsf{succ}\ t_1 \Rightarrow \mathsf{succ}\ nv = a$.
- **(SuccError)** Suppose $t_1' \Rightarrow \mathsf{error}$ and $a = \mathsf{error}$. Since $P(t_1, t_1')$, it follows that $t_1 \Rightarrow \mathsf{error}$. Thus rule (CE-SuccError) shows us that $\mathsf{succ}\ t_1 \Rightarrow \mathsf{error} = a$.

(PredBool) Similar to (SuccBool).

(PredZero) We must show that $P(\mathsf{pred}\ \mathsf{0}, \mathsf{error})$. Suppose a is an answer, and $\mathsf{error} \Rightarrow a$. We must show that $\mathsf{pred}\ \mathsf{0} \Rightarrow a$. By rule (CE-Error) and Exercise 2, we have that $a = \mathsf{error}$. By rule (CE-Zero), we have that $\mathsf{0} \Rightarrow \mathsf{0}$. Thus, by rule (CE-PredZero), we have that $\mathsf{pred}\ \mathsf{0} \Rightarrow \mathsf{error} = a$.

(PredSucc) Suppose nv is a numeric value. We must show that $P(\mathsf{pred}(\mathsf{succ}\ nv), nv)$. Suppose a is an answer, and $nv \Rightarrow a$. We must show that $\mathsf{pred}(\mathsf{succ}\ nv) \Rightarrow a$. By Lemma 3.1, we have that $nv \Rightarrow nv$. Thus, by Exercise 2, it follows that $a = nv$. By rule (CE-SuccNum), we have that $\mathsf{succ}\ nv \Rightarrow \mathsf{succ}\ nv$, so that rule (CE-PredSucc) shows us that $\mathsf{pred}(\mathsf{succ}\ nv) \Rightarrow nv = a$.

(PredError) Similar to (SuccError).

(Pred) Suppose t_1 and t_1' are terms, $t_1 \to t_1'$ and assume the inductive hypothesis, $P(t_1, t_1')$. We must show that $P(\mathsf{pred}\ t_1, \mathsf{pred}\ t_1')$. Suppose a is an answer, and $\mathsf{pred}\ t_1' \Rightarrow a$. We must show that $\mathsf{pred}\ t_1 \Rightarrow a$. Applying the inversion lemma for the complete evaluation relation to $\mathsf{pred}\ t_1' \Rightarrow a$, there are four cases to consider.

- **(PredBool)** Suppose there is a boolean value bv such that $t_1' \Rightarrow bv$ and $a = \mathsf{error}$. Since $P(t_1, t_1')$, it follows that $t_1 \Rightarrow \mathsf{error}$. Thus rule (CE-PredBool) shows us that $\mathsf{pred}\ t_1 \Rightarrow \mathsf{error} = a$.
- **(PredZero)** Suppose $t_1' \Rightarrow \mathsf{0}$ and $a = \mathsf{error}$. Since $P(t_1, t_1')$, it follows that $t_1 \Rightarrow \mathsf{0}$. Thus rule (CE-PredZero) shows us that $\mathsf{pred}\ t_1 \Rightarrow \mathsf{error} = a$.
- **(PredSucc)** Suppose there is a numeric value nv such that $t_1' \Rightarrow \mathsf{succ}\ nv$ and $a = nv$. Since $P(t_1, t_1')$, it follows that $t_1 \Rightarrow \mathsf{succ}\ nv$. Thus rule (CE-PredSucc) shows us that $\mathsf{pred}\ t_1 \Rightarrow nv = a$.
- **(PredError)** Suppose $t_1' \Rightarrow \mathsf{error}$ and $a = \mathsf{error}$. Since $P(t_1, t_1')$, it follows that $t_1 \Rightarrow \mathsf{error}$. Thus rule (CE-PredError) shows us that $\mathsf{pred}\ t_1 \Rightarrow \mathsf{error} = a$.

(IszeroBool) Similar to (SuccBool).

(IszeroZero) We must show that $P(\mathsf{iszero}\ \mathsf{0}, \mathsf{true})$. Suppose a is an answer, and $\mathsf{true} \Rightarrow a$. We must show that $\mathsf{iszero}\ \mathsf{0} \Rightarrow a$. By rule (CE-True) and Exercise 2, we have that $a = \mathsf{true}$. Thus, by rules (CE-Zero) and (CE-IszeroZero), it follows that $\mathsf{iszero}\ \mathsf{0} \Rightarrow \mathsf{true} = a$.

(IszeroSucc) Suppose nv is a numeric value. We must show that $P(\mathsf{iszero}(\mathsf{succ}\ nv), \mathsf{false})$. Suppose a is an answer and $\mathsf{false} \Rightarrow a$. We must show that $\mathsf{iszero}(\mathsf{succ}\ nv) \Rightarrow a$. By rule (CE-False) and Exercise 2, we have that $a = \mathsf{false}$. By Lemma 3.1, we have that $\mathsf{succ}\ nv \Rightarrow \mathsf{succ}\ nv$. Thus rule (CE-IszeroSucc) shows us that $\mathsf{iszero}(\mathsf{succ}\ nv) \Rightarrow \mathsf{false} = a$.

(IszeroError) Similar to (SuccError)

(Iszero) Similar to (Pred).

(OtherwiseValue) Suppose t is a term and v is a value. We must show that $P(v\ \mathsf{otherwise}\ t, v)$. Suppose a is an answer, and $v \Rightarrow a$. We must show that $v\ \mathsf{otherwise}\ t \Rightarrow a$. By Lemma 3.1, we have that $v \Rightarrow v$. Thus, by Exercise 2, it follows that $a = v$. Hence rule (CE-OtherwiseValue) shows us that $v\ \mathsf{otherwise}\ t \Rightarrow v = a$.

(OtherwiseError) Suppose t is a term. We must show that $P(\textsf{error otherwise } t, t)$. Suppose a is an answer, and $t \Rightarrow a$. We must show that $\textsf{error otherwise } t \Rightarrow a$. By rule (CE-Error), we have that $\textsf{error} \Rightarrow \textsf{error}$. Then, since $\textsf{error} \Rightarrow \textsf{error}$ and $t \Rightarrow a$, rule (CE-OtherwiseError) shows us that $\textsf{error otherwise } t \Rightarrow a$.

(Otherwise) Suppose t_1 and t_1' are terms, $t_1 \to t_1'$, and assume the inductive hypothesis, $P(t_1, t_1')$. We must show that $P(t_1 \textsf{ otherwise } t_2, t_1' \textsf{ otherwise } t_2)$. Suppose a is an answer, and $t_1' \textsf{ otherwise } t_2 \Rightarrow a$. We must show that $t_1 \textsf{ otherwise } t_2 \Rightarrow a$. Applying the inversion lemma for the complete evaluation relation to $t_1' \textsf{ otherwise } t_2 \Rightarrow a$, there are two cases to consider.

- **(OtherwiseValue)** Suppose there is a value v such that $t_1' \Rightarrow v$ and $a = v$. Since $P(t_1, t_1')$, it follows that $t_1 \Rightarrow v$. Thus rule (CE-OtherwiseValue) shows us that $t_1 \textsf{ otherwise } t_2 \Rightarrow v = a$.

- **(OtherwiseError)** Suppose $t_1' \Rightarrow \textsf{error}$ and $t_2 \Rightarrow a$. Since $P(t_1, t_1')$, it follows that $t_1 \Rightarrow \textsf{error}$. Because $t_1 \Rightarrow \textsf{error}$ and $t_2 \Rightarrow a$, rule (CE-OtherwiseError) shows us that $t_1 \textsf{ otherwise } t_2 \Rightarrow a$.

□

Lemma 3.3
For all terms t_1 and t_2 and answers a, if $t_1 \to^* t_2 \Rightarrow a$, then $t_1 \Rightarrow a$.

Proof. Define a predicate P on pairs of terms by: $P(t_1, t_2)$ iff, for all answers a, if $t_2 \Rightarrow a$, then $t_1 \Rightarrow a$. It will suffice to show that, for all terms t_1 and t_2, if $t_1 \to^* t_2$, then $P(t_1, t_2)$. (To see that this is so, suppose t_1 and t_2 are terms, a is an answer and $t_1 \to^* t_2 \Rightarrow a$. Then $P(t_1, t_2)$. But $t_2 \Rightarrow a$, and thus $t_1 \Rightarrow a$.) We proceed using the principle of induction on the reflexive-transitive closure of the evaluation relation.

(Eval) Suppose t_1 and t_2 are terms and $t_1 \to t_2$. We must show that $P(t_1, t_2)$. Suppose a is an answer and $t_2 \Rightarrow a$. We must show that $t_1 \Rightarrow a$. Because $t_1 \to t_2 \Rightarrow a$, Lemma 3.2 tells us that $t_1 \Rightarrow a$.

(Refl) Suppose t is a term. We must show that $P(t, t)$. Suppose a is an answer and $t \Rightarrow a$. Then $t \Rightarrow a$, as required.

(Trans) Suppose t_1, t_2 and t_3 are terms, $t_1 \to^* t_2$ and $t_2 \to^* t_3$, and assume the inductive hypothesis, $P(t_1, t_2)$ and $P(t_2, t_3)$. We must show that $P(t_1, t_3)$. Suppose a is an answer and $t_3 \Rightarrow a$. We must show that $t_1 \Rightarrow a$. Since $P(t_2, t_3)$ and $t_3 \Rightarrow a$, we have that $t_2 \Rightarrow a$. Then, since $P(t_1, t_2)$ and $t_2 \Rightarrow a$, we have that $t_1 \Rightarrow a$.

□

Lemma 3.4

(1) For all terms t_1, t_1', t_2 and t_3, if $t_1 \to^* t_1'$, then $\textsf{if } t_1 \textsf{ then } t_2 \textsf{ else } t_3 \to^* \textsf{if } t_1' \textsf{ then } t_2 \textsf{ else } t_3$.

(2) For all terms t_1 and t_1', if $t_1 \to^* t_1'$, then $\textsf{succ } t_1 \to^* \textsf{succ } t_1'$.

(3) For all terms t_1 and t_1', if $t_1 \to^* t_1'$, then $\textsf{pred } t_1 \to^* \textsf{pred } t_1'$.

(4) For all terms t_1 and t_1', if $t_1 \to^* t_1'$, then $\mathsf{iszero}\ t_1 \to^* \mathsf{iszero}\ t_1'$.

(5) For all terms t_1, t_1' and t_2, if $t_1 \to^* t_1'$, then $t_1\ \mathsf{otherwise}\ t_2 \to^* t_1'\ \mathsf{otherwise}\ t_2$.

Proof. We prove Part (1), the other parts being similar.

Let t_2 and t_3 be terms. We must show that, for all terms t_1 and t_1', if $t_1 \to^* t_1'$, then $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to^* \mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$. Define a predicate P on pairs of terms by: $P(t_1, t_1')$ iff $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to^* \mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$. Thus it will suffice to show that, for all terms t_1 and t_1', if $t_1 \to^* t_1'$, then $P(t_1, t_1')$. We proceed using the principle of induction on the reflexive-transitive closure of the evaluation relation.

(Eval) Suppose t_1 and t_1' are terms and $t_1 \to t_1'$. We must show that $P(t_1, t_1')$, i.e., $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to^* \mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$. By rule (E-If), we have that $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to \mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$. Thus $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to^* \mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$ follows by rule (RTCE-Eval).

(Refl) Suppose t_1 is a term. We must show that $P(t_1, t_1)$, i.e., $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to^* \mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$. And this follows by rule (RTCE-Refl).

(Trans) Suppose t_1, t_1' and t_1'' are terms, $t_1 \to^* t_1'$ and $t_1' \to^* t_1''$, and assume the inductive hypothesis, $P(t_1, t_1')$ and $P(t_1', t_1'')$. We must show that $P(t_1, t_1'')$. Thus we have that $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to^* \mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$ and $\mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to^* \mathsf{if}\ t_1''\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$, so that, by rule (RTCE-Trans), it follows that $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to^* \mathsf{if}\ t_1''\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$, i.e., $P(t_1, t_1'')$.

□

Lemma 3.5
For all terms t and answers a, if $t \Rightarrow a$, then $t \to^* a$.

Proof. Define a predicate P between a term and an answer: $P(t, a)$ iff $t \to^* a$. It will suffice to show that, for all terms t and answers a, if $t \Rightarrow a$, then $P(t, a)$. We proceed using the principle of induction on the complete evaluation relation.

(True) We must show that $P(\mathsf{true}, \mathsf{true})$, i.e., $\mathsf{true} \to^* \mathsf{true}$, and this follows by rule (RTCE-Refl).

(False) Similar to (True).

(Zero) Similar to (True).

(Error) Similar to (True).

(IfTrue) Suppose t_1, t_2 and t_3 are terms, a is an answer, $t_1 \Rightarrow \mathsf{true}$ and $t_2 \Rightarrow a$, and assume the inductive hypothesis, $P(t_1, \mathsf{true})$ and $P(t_2, a)$. We must show that $P(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3, a)$. Because $P(t_1, \mathsf{true})$ and $P(t_2, a)$, we have that $t_1 \to^* \mathsf{true}$ and $t_2 \to^* a$. Thus, by Lemma 3.4(1) and rule (E-IfTrue), we have that

$$\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to^* \mathsf{if}\ \mathsf{true}\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to t_2 \to^* a.$$

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to^* a$, i.e., $P(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3, a)$.

(IfFalse) Similar to (IfTrue), but using (E-IfFalse).

(IfNum) Suppose t_1, t_2 and t_3 are terms, nv is a numeric value and $t_1 \Rightarrow nv$, and assume the inductive hypothesis, $P(t_1, nv)$. We must show that $P(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3, \mathsf{error})$. Because $P(t_1, nv)$, we have that $t_1 \rightarrow^* nv$. Thus, by Lemma 3.4(1) and rule (E-IfNum), we have that

$$\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \rightarrow^* \mathsf{if}\ nv\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \rightarrow \mathsf{error}.$$

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \rightarrow^* \mathsf{error}$, i.e., $P(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3, \mathsf{error})$.

(IfError) Suppose t_1, t_2 and t_3 are terms and $t_1 \Rightarrow \mathsf{error}$, and assume the inductive hypothesis, $P(t_1, \mathsf{error})$. We must show that $P(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3, \mathsf{error})$. Because $P(t_1, \mathsf{error})$, we have that $t_1 \rightarrow^* \mathsf{error}$. Thus, by Lemma 3.4(1) and rule (E-IfError), we have that

$$\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \rightarrow^* \mathsf{if}\ \mathsf{error}\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \rightarrow \mathsf{error}.$$

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \rightarrow^* \mathsf{error}$, i.e., $P(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3, \mathsf{error})$.

(SuccBool) Suppose t is a term, bv is a boolean value and $t \Rightarrow bv$, and assume the inductive hypothesis, $P(t, bv)$. We must show that $P(\mathsf{succ}\ t, \mathsf{error})$. Because $P(t, bv)$, we have that $t \rightarrow^* bv$. Thus, by Lemma 3.4(2) and rule (E-SuccBool), we have that

$$\mathsf{succ}\ t \rightarrow^* \mathsf{succ}\ bv \rightarrow \mathsf{error}.$$

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that $\mathsf{succ}\ t \rightarrow^* \mathsf{error}$, i.e., $P(\mathsf{succ}\ t, \mathsf{error})$.

(SuccNum) Suppose t is a term, nv is a numeric value and $t \Rightarrow nv$, and assume the inductive hypothesis, $P(t, nv)$. We must show that $P(\mathsf{succ}\ t, \mathsf{succ}\ nv)$. Because $P(t, nv)$, we have that $t \rightarrow^* nv$. Thus, by Lemma 3.4(2), we have that $\mathsf{succ}\ t \rightarrow^* \mathsf{succ}\ nv$, i.e., $P(\mathsf{succ}\ t, \mathsf{succ}\ nv)$.

(SuccError) Suppose t is a term and $t \Rightarrow \mathsf{error}$, and assume the inductive hypothesis, $P(t, \mathsf{error})$. We must show that $P(\mathsf{succ}\ t, \mathsf{error})$. Because $P(t, \mathsf{error})$, we have that $t \rightarrow^* \mathsf{error}$. Thus, by Lemma 3.4(2) and rule (E-SuccError), we have that

$$\mathsf{succ}\ t \rightarrow^* \mathsf{succ}\ \mathsf{error} \rightarrow \mathsf{error}.$$

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that $\mathsf{succ}\ t \rightarrow^* \mathsf{error}$, i.e., $P(\mathsf{succ}\ t, \mathsf{error})$.

(PredBool) Similar to (SuccBool).

(PredZero) Suppose t is a term and $t \Rightarrow \mathsf{0}$, and assume the inductive hypothesis, $P(t, \mathsf{0})$. We must show that $P(\mathsf{pred}\ t, \mathsf{error})$. Because $P(t, \mathsf{0})$, we have that $t \rightarrow^* \mathsf{0}$. Thus, by Lemma 3.4(3) and rule (E-PredZero), we have that

$$\mathsf{pred}\ t \rightarrow^* \mathsf{pred}\ \mathsf{0} \rightarrow \mathsf{error}.$$

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that $\mathsf{pred}\ t \rightarrow^* \mathsf{error}$, i.e., $P(\mathsf{pred}\ t, \mathsf{error})$.

(PredSucc) Suppose t is a term, nv is a numeric value and $t \Rightarrow \mathsf{succ}\ nv$, and assume the inductive hypothesis, $P(t, \mathsf{succ}\ nv)$. We must show that $P(\mathsf{pred}\ t, nv)$. Because $P(t, \mathsf{succ}\ nv)$, we have that $t \rightarrow^* \mathsf{succ}\ nv$. Thus, by Lemma 3.4(3) and rule (E-PredSucc), we have that

$$\mathsf{pred}\ t \rightarrow^* \mathsf{pred}(\mathsf{succ}\ nv) \rightarrow nv.$$

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that $\mathsf{pred}\ t \rightarrow^* nv$, i.e., $P(\mathsf{pred}\ t, nv)$.

(PredError) Similar to (SuccError).

(IszeroBool) Similar to (SuccBool).

(IszeroZero) Similar to (PredZero).

(IszeroSucc) Similar to (PredSucc).

(IszeroError) Similar to (SuccError).

(OtherwiseValue) Suppose t_1 and t_2 are terms, v is a value and $t_1 \Rightarrow v$, and assume the inductive hypothesis, $P(t_1, v)$. We must show that $P(t_1\ \mathsf{otherwise}\ t_2, v)$. Because $P(t_1, v)$, we have that $t_1 \rightarrow^* v$. Thus, by Lemma 3.4(5) and rule (E-OtherwiseValue), we have that

$$t_1\ \mathsf{otherwise}\ t_2 \rightarrow^* v\ \mathsf{otherwise}\ t_2 \rightarrow v.$$

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that $t_1\ \mathsf{otherwise}\ t_2 \rightarrow^* v$, i.e., $P(t_1\ \mathsf{otherwise}\ t_2, v)$.

(OtherwiseError) Suppose t_1 and t_2 are terms, a is an answer, $t_1 \Rightarrow \mathsf{error}$ and $t_2 \Rightarrow a$, and assume the inductive hypothesis, $P(t_1, \mathsf{error})$ and $P(t_2, a)$. We must show that $P(t_1\ \mathsf{otherwise}\ t_2, a)$. Because $P(t_1, \mathsf{error})$ and $P(t_2, a)$, we have that $t_1 \rightarrow^* \mathsf{error}$ and $t_2 \rightarrow^* a$. Thus, by Lemma 3.4(5) and rule (E-OtherwiseError), we have that

$$t_1\ \mathsf{otherwise}\ t_2 \rightarrow^* \mathsf{error}\ \mathsf{otherwise}\ t_2 \rightarrow t_2 \rightarrow^* a.$$

Hence, using (RTCE-Eval) and (RTCE-Trans), we have that $t_1\ \mathsf{otherwise}\ t_2 \rightarrow^* a$, i.e., $P(t_1\ \mathsf{otherwise}\ t_2, a)$.

□

To prove that $\rightsquigarrow = \Rightarrow$, it will suffice to show that, for all terms t and answers a, $t \rightsquigarrow a$ iff $t \Rightarrow a$. Suppose t is a term and a is an answer.

- Suppose $t \rightsquigarrow a$. Then $t \rightarrow^* a$. By Lemma 3.1, we have that $a \Rightarrow a$. Thus $t \rightarrow^* a \Rightarrow a$, so that $t \Rightarrow a$, by Lemma 3.3.
- Suppose $t \Rightarrow a$. Lemma 3.5 shows us that $t \rightarrow^* a$, so that $t \rightsquigarrow a$.