CIS 705 — Programming Languages — Spring 2010

Assignment 4

Model Answers

Exercise 1

See `sum-range-untyped.txt`.

Exercise 2

Let sumRange be the term from Exercise 1. It is easy to see that it is closed. We carry out the definition of 15 closed values, in order:

(1) $\mathsf{tru} = \lambda x.\, \lambda y.\, x$;

(2) $\mathsf{fls} = \lambda x.\, \lambda y.\, y$;

(3) $\mathsf{if} = \lambda b.\, \lambda f1.\, \lambda f2.\, b\, f1\, f2\, \mathsf{tru}$;

(4) $\mathsf{fix} = \lambda f.\, \lambda y.\, (\lambda x.\, f(\lambda y.\, x\, x\, y))\, (\lambda x.\, f(\lambda y.\, x\, x\, y))\, y = \lambda f.\, \lambda y.\, N_f\, N_f\, y$, where (as in the lecture notes) $N_t = \lambda x.\, t\, (\lambda y.\, x\, x\, y)$, when t is a closed term or the variable f;

(5) $\mathsf{zero} = \lambda s.\, \lambda z.\, z$;

(6) $\mathsf{one} = \lambda s.\, \lambda z.\, s\, z$;

(7) $\mathsf{plus} = \lambda n.\, \lambda m.\, \lambda s.\, \lambda z.\, n\, s\, (m\, s\, z)$;

(8) $\mathsf{isZero} = \lambda n.\, n\, (\lambda x.\, \mathsf{fls})\, \mathsf{tru}$;

(9) $\mathsf{pair} = \lambda f.\, \lambda s.\, \lambda b.\, b\, f\, s$;

(10) $\mathsf{fst} = \lambda p.\, p\, \mathsf{tru}$;

(11) $\mathsf{snd} = \lambda p.\, p\, \mathsf{fls}$;

(12) $\mathsf{pred} = \lambda n.$
$$\begin{array}{l} \mathsf{fst}(n \\ \quad (\lambda p.\, \mathsf{pair}\, (\mathsf{snd}\, p)\, (\mathsf{plus}\, \mathsf{one}\, (\mathsf{snd}\, p))) \\ \quad (\mathsf{pair}\, \mathsf{zero}\, \mathsf{zero})); \end{array}$$

(13) $\mathsf{minus} = \lambda n.\, \lambda m.\, m\, \mathsf{pred}\, n$;

(14) $\mathsf{lessThanOrEqualTo} = \lambda n.\, \lambda m.\, \mathsf{isZero}(\mathsf{minus}\, n\, m)$;

(15) $\mathsf{sumRangeBody} = \lambda \mathit{sumRange}.\, \lambda n.\, \lambda m.$
$$\begin{array}{l} \mathsf{if}\, (\mathsf{lessThanOrEqualTo}\, n\, m) \\ \quad (\lambda x.\, \mathsf{plus}\, n\, (\mathit{sumRange}\, (\mathsf{plus}\, n\, \mathsf{one})\, m)) \\ \quad (\lambda x.\, \mathsf{zero}). \end{array}$$

Lemma 2.1
$\mathsf{sumRange} \to^* \mathsf{fix}\,\mathsf{sumRangeBody}$.

Proof. Let's write x_1, ..., x_{15} for the variables *tru*, *fls*, *if*, *fix*, *zero*, *one*, *plus*, *isZero*, *pair*, *fst*, *snd*, *pred*, *minus*, *lessThanOrEqualTo* and *sumRangeBody*, respectively (note that these are the same names as the left sides of the above definitions), and let's write $v_1, \ldots, v_{15}$ for the closed values tru, fls, if, fix, zero, one, plus, isZero, pair, fst, snd, pred, minus, lessThanOrEqualTo and sumRangeBody, respectively.

We write $\mathsf{sumRange}_i$, for $0 \leq i \leq 15$, for the term whose free variables are included in x_1, ..., x_i that is found by going $2i$ levels down the leftmost path in (the tree) sumRange (so that $\mathsf{sumRange}_0 = \mathsf{sumRange}$, and $\mathsf{sumRange}_{15} = \mathit{fix}\,\mathit{sumRangeBody}$.)

For $0 \leq i \leq 15$, define the closed term

$$\mathsf{sumRangeSub}_i = [x_1 \mapsto v_1] \cdots [x_i \mapsto v_i]\,\mathsf{sumRange}_i,$$

so that $\mathsf{sumRange} = \mathsf{sumRange}_0 = \mathsf{sumRangeSub}_0$ and $\mathsf{sumRangeSub}_{15} = \mathsf{fix}\,\mathsf{sumRangeBody}$. For $0 \leq i < 15$,

$$\mathsf{sumRangeSub}_i = (\lambda x_{i+1}.\,[x_1 \mapsto v_1] \cdots [x_i \mapsto v_i]\,\mathsf{sumRange}_{i+1})v_{i+1},$$

and we have that $\mathsf{sumRangeSub}_i \to \mathsf{sumRangeSub}_{i+1}$. Thus

$$\mathsf{sumRange} = \mathsf{sumRangeSub}_0 \to^{15} \mathsf{sumRangeSub}_{15} = \mathsf{fix}\,\mathsf{sumRangeBody},$$

so that $\mathsf{sumRange} \to^* \mathsf{fix}\,\mathsf{sumRangeBody}$. $\square$

From the lectures notes, we know that tru, fls, zero and one are closed values representing true, false, 0 and 1, respectively.

Given $n, m \in \mathbb{N}$, we write $+[n : m] \in \mathbb{N}$ for the sum of

$$\{\, i \in \mathbb{N} \mid n \leq i \text{ and } i \leq m \,\}.$$

Given $n, m \in \mathbb{N}$, we define $n -_{\mathbb{N}} m \in \mathbb{N}$ by:

$$n -_{\mathbb{N}} m = \begin{cases} n - m, & \text{if } m \leq n, \\ 0, & \text{if } m > n. \end{cases}$$

Lemma 2.2
For all closed terms t and $n \in \mathbb{N}$, if t converges to a closed value representing n, then:

- *if $n = 0$, then $\mathsf{isZero}\,t$ converges to a closed value representing true;*
- *if $n > 0$, then $\mathsf{isZero}\,t$ converges to a closed value representing false.*

Proof. Suppose t is a closed term, $n \in \mathbb{N}$, and t converges to a closed value representing n. Because isZero is a value, we have that

$$\mathsf{isZero}\,t \to^* \mathsf{isZero}\,\overline{t} \to \overline{t}\,(\lambda x.\,\mathsf{fls})\,\mathsf{tru}.$$

To see that $(\lambda x.\,\mathsf{fls}, \mathsf{tru}) \in \mathsf{SZ}$, we use ordinary induction to prove that, for all $n \in \mathbb{N}$, $(\lambda x.\,\mathsf{fls})^n(\mathsf{tru})$ converges.

(Basis Step) We have that $(\lambda x.\,\mathsf{fls})^0(\mathsf{tru}) = \mathsf{tru}$ converges.

(Inductive Step) Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis, $(\lambda x.\,\mathsf{fls})^n(\mathsf{tru})$ converges. Thus

$$(\lambda x.\,\mathsf{fls})^{n+1}(\mathsf{tru}) = (\lambda x.\,\mathsf{fls})((\lambda x.\,\mathsf{fls})^n(\mathsf{tru})) \to^* (\lambda x.\,\mathsf{fls})\,\overline{(\lambda x.\,\mathsf{fls})^n(\mathsf{tru})} \to \mathsf{fls},$$

so that $(\lambda x.\,\mathsf{fls})^{n+1}(\mathsf{tru})$ converges.

There are two parts to prove.

- Suppose $n = 0$. Since $\bar{t}$ represents 0, we have that $\bar{t}\,(\lambda x.\,\mathsf{fls})\,\mathsf{tru} \to^* \overline{(\lambda x.\,\mathsf{fls})^0(\mathsf{tru})} = \overline{\mathsf{tru}} = \mathsf{tru}$. Hence $\mathsf{isZero}\,t$ converges to a closed value representing true.
- Suppose $n > 0$. Since $\bar{t}$ represents n, Proposition F tells us that

$$\begin{aligned}\bar{t}\,(\lambda x.\,\mathsf{fls})\,\mathsf{tru} \to^* \overline{(\lambda x.\,\mathsf{fls})^n(\mathsf{tru})} &= \overline{(\lambda x.\,\mathsf{fls})((\lambda x.\,\mathsf{fls})^{n-1}(\mathsf{tru}))} = \overline{\overline{(\lambda x.\,\mathsf{fls})}\;\overline{(\lambda x.\,\mathsf{fls})^{n-1}(\mathsf{tru})}} \\ &= \overline{(\lambda x.\,\mathsf{fls})\,\overline{(\lambda x.\,\mathsf{fls})^{n-1}(\mathsf{tru})}} = \mathsf{fls}.\end{aligned}$$

  Hence $\mathsf{isZero}\,t$ converges to a closed value representing false.

□

Lemma 2.3
For all closed terms t and $n \in \mathbb{N}$, if t converges to a closed value representing n, then $\mathsf{pred}\,t$ converges to a closed value representing $n -_{\mathbb{N}} 1$.

Proof. Suppose t is a closed term, $n \in \mathbb{N}$ and t converges to a closed value representing n. Define closed terms ss and zz by:

$$\begin{aligned}\mathsf{ss} &= \lambda p.\,\mathsf{pair}\,(\mathsf{snd}\,p)\,(\mathsf{plus}\,\mathsf{one}\,(\mathsf{snd}\,p)),\\ \mathsf{zz} &= \mathsf{pair}\,\mathsf{zero}\,\mathsf{zero}.\end{aligned}$$

Because pred is a value, we have that $\mathsf{pred}\,t \to^* \mathsf{pred}\,\bar{t} \to \mathsf{fst}(\bar{t}\,\mathsf{ss}\,\mathsf{zz})$.

We say that a closed value v *represents* a pair (n, m) of natural numbers iff there are closed values v_1 and v_2 such that v represents (v_1, v_2), v_1 represents n, and v_2 represents m. By Proposition H, we have that zz converges to a closed value representing $(0, 0)$. Since fst is a value, $\mathsf{fst}(\bar{t}\,\mathsf{ss}\,\mathsf{zz}) \to^* \mathsf{fst}(\bar{t}\,\mathsf{ss}\,\overline{\mathsf{zz}})$. Thus $\mathsf{pred}\,t \to^* \mathsf{fst}(\bar{t}\,\mathsf{ss}\,\overline{\mathsf{zz}})$.

We use ordinary induction to prove that, for all $m \in \mathbb{N}$, $\mathsf{ss}^m(\overline{\mathsf{zz}})$ converges to a closed value representing $(m -_{\mathbb{N}} 1, m)$ (so that $(\mathsf{ss}, \overline{\mathsf{zz}}) \in \mathsf{SZ}$).

Basis Step We have that $\mathsf{ss}^0(\overline{\mathsf{zz}}) = \overline{\mathsf{zz}}$, which represents $(0, 0) = (0 -_{\mathbb{N}} 1, 0)$. Thus $\mathsf{ss}^0(\overline{\mathsf{zz}})$ converges to a closed value representing $(0 -_{\mathbb{N}} 1, 0)$.

Inductive Step Suppose $m \in \mathbb{N}$, and assume the inductive hypothesis, $\mathsf{ss}^m(\overline{\mathsf{zz}})$ converges to a closed value representing $(m -_{\mathbb{N}} 1, m)$. We must show that $\mathsf{ss}^{m+1}(\overline{\mathsf{zz}})$ converges to a closed value representing $((m + 1) -_{\mathbb{N}} 1, m + 1)$. Let u be the closed value such that $\mathsf{ss}^m(\overline{\mathsf{zz}}) \to^* u$,

$\mathsf{fst}\, u$ converges to a closed value representing $m -_{\mathbb{N}} 1$, and $\mathsf{snd}\, u$ converges to a closed value representing m. Since ss is a value,

$$\begin{aligned}\mathsf{ss}^{m+1}(\overline{\mathsf{zz}}) = &\ \mathsf{ss}(\mathsf{ss}^m(\overline{\mathsf{zz}})) \\ \to^* &\ \mathsf{ss}\, u \\ \to &\ \mathsf{pair}\,(\mathsf{snd}\, u)\,(\mathsf{plus}\,\mathsf{one}\,(\mathsf{snd}\, u)).\end{aligned}$$

Since one is a closed value representing 1, and $\mathsf{snd}\, u$ converges to a closed value representing m, Proposition J tells us that $\mathsf{plus}\,\mathsf{one}\,(\mathsf{snd}\, u)$ converges to a closed value representing $m+1$. Thus, by Proposition H, $\mathsf{pair}\,(\mathsf{snd}\, u)\,(\mathsf{plus}\,\mathsf{one}\,(\mathsf{snd}\, u))$ converges to a closed value representing $(m, m+1)$. Thus $\mathsf{ss}^{m+1}(\overline{\mathsf{zz}})$ converges to a closed value representing $(m, m+1)$. So it remains to show that $(m+1) -_{\mathbb{N}} 1 = m$, and this follows since $1 \leq m+1$, and thus $(m+1) -_{\mathbb{N}} 1 = (m+1) - 1 = m$.

Thus $\bar{t}\,\mathsf{ss}\,\overline{\mathsf{zz}}$ converges to $\overline{\mathsf{ss}^n(\overline{\mathsf{zz}})}$, which is a closed value representing $(n -_{\mathbb{N}} 1, n)$. Hence $\mathsf{fst}\,\overline{\mathsf{ss}^n(\overline{\mathsf{zz}})}$ converges to a closed value representing $n -_{\mathbb{N}} 1$. Since fst is a value, it follows that $\mathsf{fst}(\bar{t}\,\mathsf{ss}\,\overline{\mathsf{zz}})$ converges to a closed value representing $n -_{\mathbb{N}} 1$, so that $\mathsf{pred}\, t$ converges to a closed value representing $n -_{\mathbb{N}} 1$. □

Lemma 2.4
For all closed terms t and t' and $n, m \in \mathbb{N}$, if t converges to a closed value representing n, and t' converges to a closed value representing m, then $\mathsf{minus}\, t\, t'$ converges to a closed value representing $n -_{\mathbb{N}} m$.

Proof. Since minus is a value, by Proposition G, it will suffice to show that, for all closed values v and v', and $n, m \in \mathbb{N}$, if v represents n, and v' represents m, then $\mathsf{minus}\, v\, v'$ converges to a closed value representing $n -_{\mathbb{N}} m$. (To see this, suppose t and t' are closed terms, $n, m \in \mathbb{N}$, t converges to a closed value representing n, and t' converges to a closed value representing m. Thus $\overline{\mathsf{minus}}\,\bar{t}\,\overline{t'} = \mathsf{minus}\,\bar{t}\,\overline{t'}$ converges to a closed value representing $n -_{\mathbb{N}} m$. Hence, by Proposition G, $\mathsf{minus}\, t\, t'$ converges and

$$\overline{\mathsf{minus}\, t\, t'} = \overline{\overline{\mathsf{minus}}\,\bar{t}\,\overline{t'}},$$

and thus $\mathsf{minus}\, t\, t'$ converges to a closed value representing $n -_{\mathbb{N}} m$.) Suppose v and v' are closed values, $n, m \in \mathbb{N}$, v represents n, and v' represents m. We have that $\mathsf{minus}\, v\, v' \to^* v'\, \mathsf{pred}\, v$.

We use ordinary induction to prove that, for all $l \in \mathbb{N}$, $\mathsf{pred}^l(v)$ converges to a closed value representing $n -_{\mathbb{N}} l$ (so that $(\mathsf{pred}, v) \in \mathsf{SZ}$).

- **(Basis Step)** We have that $\mathsf{pred}^0(v) = v$ represents $n = n -_{\mathbb{N}} 0$, so that $\mathsf{pred}^0(v)$ converges to a closed value representing $n -_{\mathbb{N}} 0$.
- **(Inductive Step)** Suppose $l \in \mathbb{N}$, and assume the inductive hypothesis, $\mathsf{pred}^l(v)$ converges to a closed value representing $n -_{\mathbb{N}} l$. We must show that $\mathsf{pred}^{l+1}(v)$ converges to a closed value representing $n -_{\mathbb{N}} (l+1)$. By the inductive hypothesis and Lemma 2.3, we have that $\mathsf{pred}^{l+1}(v) = \mathsf{pred}(\mathsf{pred}^l(v))$ converges to a closed value representing $(n -_{\mathbb{N}} l) -_{\mathbb{N}} 1$. So it remains to show that $(n -_{\mathbb{N}} l) -_{\mathbb{N}} 1 = n -_{\mathbb{N}} (l+1)$. There are two cases to consider.
  - Suppose $l + 1 \leq n$. Then $l \leq n$ and $1 \leq n - l$, so that $(n -_{\mathbb{N}} l) -_{\mathbb{N}} 1 = (n - l) -_{\mathbb{N}} 1 = (n - l) - 1 = n - (l+1) = n -_{\mathbb{N}} (l+1)$.

- Suppose $l+1 > n$. Thus $n -_{\mathbb{N}} (l+1) = 0$, so it will suffice to show that $(n -_{\mathbb{N}} l) -_{\mathbb{N}} 1 = 0$. If $l > n$, then $(n -_{\mathbb{N}} l) -_{\mathbb{N}} 1 = 0 -_{\mathbb{N}} 1 = 0$. Otherwise, $l = n$, so that $(n -_{\mathbb{N}} l) -_{\mathbb{N}} 1 = 0 -_{\mathbb{N}} 1 = 0$.

Thus we have that $v' \, \mathsf{pred} \, v$ converges to $\overline{\mathsf{pred}^m(v)}$, which is a closed value representing $n -_{\mathbb{N}} m$. Hence $\mathsf{minus} \, v \, v'$ converges to a closed value representing $n -_{\mathbb{N}} m$. □

Lemma 2.5
For all closed terms t and t' and $n, m \in \mathbb{N}$, if t converges to a closed value representing n, and t' converges to a closed value representing m, then:

- *if $n \leq m$, then $\mathsf{lessThanOrEqualTo} \, t \, t'$ converges to a closed value representing true;*
- *if $n > m$, then $\mathsf{lessThanOrEqualTo} \, t \, t'$ converges to a closed value representing false.*

Proof. By Proposition G, it will suffice to show that, for all closed values v and v', and $n, m \in \mathbb{N}$, if v represents n, and v' represents m, then:

- if $n \leq m$, then $\mathsf{lessThanOrEqualTo} \, v \, v'$ converges to a closed value representing true;
- if $n > m$, then $\mathsf{lessThanOrEqualTo} \, v \, v'$ converges to a closed value representing false.

Suppose v and v' are closed values, $n, m \in \mathbb{N}$, v represents n, and v' represents m. We have that

$$\mathsf{lessThanOrEqualTo} \, v \, v' \to^* \mathsf{isZero}(\mathsf{minus} \, v \, v').$$

By Lemma 2.4, we have that $\mathsf{minus} \, v \, v'$ converges to a closed value representing $n -_{\mathbb{N}} m$. There are two parts to prove.

- Suppose $n \leq m$. Thus $m \geq n$, so that $n -_{\mathbb{N}} m = 0$. Since $\mathsf{minus} \, v \, v'$ converges to a closed value representing 0, Lemma 2.2 tells us that $\mathsf{isZero}(\mathsf{minus} \, v \, v')$ converges to a closed value representing true. Thus $\mathsf{lessThanOrEqualTo} \, v \, v'$ converges to a closed value representing true.
- Suppose $n > m$. Thus $m < n$, so that $n -_{\mathbb{N}} m = n - m > 0$. Since $\mathsf{minus} \, v \, v'$ converges to a closed value representing a non-zero natural number, Lemma 2.2 tells us that $\mathsf{isZero}(\mathsf{minus} \, v \, v')$ converges to a closed value representing false. Thus $\mathsf{lessThanOrEqualTo} \, v \, v'$ converges to a closed value representing false.

□

Let $\mathsf{sumRangeBodyFix}$ be the closed value

$$\lambda y. \, N_{\mathsf{sumRangeBody}} \, N_{\mathsf{sumRangeBody}} \, y.$$

Lemma 2.6
(1) $\mathsf{sumRange} \to^* \mathsf{sumRangeBodyFix}$.

(2) For all closed values v, $\mathsf{sumRangeBodyFix} \, v \to^* \mathsf{sumRangeBody} \, \mathsf{sumRangeBodyFix} \, v$.

Proof.

(1) By Lemma 2.1, we have that

$$\begin{aligned}\mathsf{sumRange} &\to^* \mathsf{fix}\,\mathsf{sumRangeBody} \\ &\to \lambda y.\, N_{\mathsf{sumRangeBody}}\, N_{\mathsf{sumRangeBody}}\, y \\ &= \mathsf{sumRangeBodyFix}.\end{aligned}$$

(2) Suppose v is a closed value. Then

$$\begin{aligned}\mathsf{sumRangeBodyFix}\, v &\to N_{\mathsf{sumRangeBody}}\, N_{\mathsf{sumRangeBody}}\, v \\ &= (\lambda x.\, \mathsf{sumRangeBody}\,(\lambda y.\, x\, x\, y))\, N_{\mathsf{sumRangeBody}}\, v \\ &\to \mathsf{sumRangeBody}\,(\lambda y.\, N_{\mathsf{sumRangeBody}}\, N_{\mathsf{sumRangeBody}}\, y)\, v \\ &= \mathsf{sumRangeBody}\,\mathsf{sumRangeBodyFix}\, v.\end{aligned}$$

□

Lemma 2.7
For all closed values v and v', and $n, m \in \mathbb{N}$, if v represents n, and v' represents m, then:

- *if $n \le m$, then*

$$\mathsf{sumRangeBodyFix}\, v\, v' \to^* \mathsf{plus}\, v\, (\mathsf{sumRangeBodyFix}\,(\mathsf{plus}\, v\, \mathsf{one})\, v');$$

- *if $n > m$, then*

$$\mathsf{sumRangeBodyFix}\, v\, v' \to^* \mathsf{zero}.$$

Proof. Suppose v and v' are closed values, $n, m \in \mathbb{N}$, v represents n, and v' represents m. By Lemma 2.6(2), we have that $\mathsf{sumRangeBodyFix}\, v\, v' \to^* \mathsf{sumRangeBody}\,\mathsf{sumRangeBodyFix}\, v\, v'$, which is related by $\to^*$ to

$$\begin{aligned}&\mathsf{if}\,(\mathsf{lessThanOrEqualTo}\, v\, v') \\ &\quad (\lambda x.\, \mathsf{plus}\, v\,(\mathsf{sumRangeBodyFix}\,(\mathsf{plus}\, v\, \mathsf{one})\, v')) \\ &\quad (\lambda x.\, \mathsf{zero}).\end{aligned}$$

There are two parts to prove.

- Suppose $n \le m$. By Lemma 2.5, we have that $\mathsf{lessThanOrEqualTo}\, v\, v' \to^* u$, where u is a closed value representing true. Thus

$$\begin{aligned}&\mathsf{if}\,(\mathsf{lessThanOrEqualTo}\, v\, v') \\ &\quad (\lambda x.\, \mathsf{plus}\, v\,(\mathsf{sumRangeBodyFix}\,(\mathsf{plus}\, v\, \mathsf{one})\, v')) \\ &\quad (\lambda x.\, \mathsf{zero})\end{aligned}$$

  is related by $\to^*$ to

$$\begin{aligned}&\mathsf{if}\, u \\ &\quad (\lambda x.\, \mathsf{plus}\, v\,(\mathsf{sumRangeBodyFix}\,(\mathsf{plus}\, v\, \mathsf{one})\, v')) \\ &\quad (\lambda x.\, \mathsf{zero}),\end{aligned}$$

which is related by $\to^*$ to

$$u\,(\lambda x.\,\mathsf{plus}\,v\,(\mathsf{sumRangeBodyFix}\,(\mathsf{plus}\,v\,\mathsf{one})\,v'))\,(\lambda x.\,\mathsf{zero})\,\mathsf{tru},$$

which (because u represents true) is related by $\to^*$ to

$$(\lambda x.\,\mathsf{plus}\,v\,(\mathsf{sumRangeBodyFix}\,(\mathsf{plus}\,v\,\mathsf{one})\,v'))\,\mathsf{tru},$$

which evaluates to

$$\mathsf{plus}\,v\,(\mathsf{sumRangeBodyFix}\,(\mathsf{plus}\,v\,\mathsf{one})\,v').$$

Thus

$$\mathsf{sumRangeBodyFix}\,v\,v' \to^* \mathsf{plus}\,v\,(\mathsf{sumRangeBodyFix}\,(\mathsf{plus}\,v\,\mathsf{one})\,v').$$

- Suppose $n > m$. By Lemma 2.5, we have that $\mathsf{lessThanOrEqualTo}\,v\,v' \to^* u$, where u is a closed value representing false. Thus

$$\begin{array}{l}\mathsf{if}\,(\mathsf{lessThanOrEqualTo}\,v\,v')\\ \quad(\lambda x.\,\mathsf{plus}\,v\,(\mathsf{sumRangeBodyFix}\,(\mathsf{plus}\,v\,\mathsf{one})\,v'))\\ \quad(\lambda x.\,\mathsf{zero})\end{array}$$

  is related by $\to^*$ to

$$\begin{array}{l}\mathsf{if}\,u\\ \quad(\lambda x.\,\mathsf{plus}\,v\,(\mathsf{sumRangeBodyFix}\,(\mathsf{plus}\,v\,\mathsf{one})\,v'))\\ \quad(\lambda x.\,\mathsf{zero}),\end{array}$$

  which is related by $\to^*$ to

$$u\,(\lambda x.\,\mathsf{plus}\,v\,(\mathsf{sumRangeBodyFix}\,(\mathsf{plus}\,v\,\mathsf{one})\,v'))\,(\lambda x.\,\mathsf{zero})\,\mathsf{tru},$$

  which (because u represents false) is related by $\to^*$ to

$$(\lambda x.\,\mathsf{zero})\,\mathsf{tru},$$

  which evaluates to zero. Thus

$$\mathsf{sumRangeBodyFix}\,v\,v' \to^* \mathsf{zero}.$$

□

Lemma 2.8
For all closed values v and v', and $n, m \in \mathbb{N}$, if v represents n, and v' represents m, then

$$\mathsf{sumRangeBodyFix}\,v\,v'$$

converges to a closed value representing $+[n : m]$.

Proof. Define a predicate $P(l)$ on $\mathbb{N}$ by: for all closed values v and v', and $n, m \in \mathbb{N}$, if $m -_{\mathbb{N}} n = l$, v represents n, and v' represents m, then

$$\mathsf{sumRangeBodyFix}\, v\, v'$$

converges to a closed value representing $+[n : m]$. It will suffice to show using ordinary induction that, for all $l \in \mathbb{N}$, $P(l)$. (To see this, suppose v and v' are closed values, $n, m \in \mathbb{N}$, v represents n, and v' represents m. Then $P(m -_{\mathbb{N}} n)$, so that

$$\mathsf{sumRangeBodyFix}\, v\, v'$$

converges to a closed value representing $+[n : m]$.)

(Basis Step) We must show $P(0)$. Suppose v and v' are closed values, $n, m \in \mathbb{N}$, $m -_{\mathbb{N}} n = 0$, v represents n, and v' represents m. We must show that

$$\mathsf{sumRangeBodyFix}\, v\, v'$$

converges to a closed value representing $+[n : m]$. Since $m -_{\mathbb{N}} n = 0$, there are two cases to consider.

- Suppose $n = m$. Then $+[n : m] = n$. Since $n \leq m$, Lemma 2.7 tells us that $\mathsf{sumRangeBodyFix}\, v\, v' \rightarrow^* \mathsf{plus}\, v\, (\mathsf{sumRangeBodyFix}\, (\mathsf{plus}\, v\, \mathsf{one})\, v')$. By Proposition J, we have that $\mathsf{plus}\, v\, \mathsf{one} \rightarrow^* u$, for a closed value u representing $n + 1$. Because u and v' are closed values representing $n + 1$ and m, respectively, and $n + 1 > m$, Lemma 2.7 tells us that $\mathsf{sumRangeBodyFix}\, u\, v' \rightarrow^* \mathsf{zero}$. Thus $\mathsf{sumRangeBodyFix}\, (\mathsf{plus}\, v\, \mathsf{one})\, v' \rightarrow^* \mathsf{zero}$, by Proposition G. Since v and zero are closed values representing n and 0, respectively, Proposition J tells us that $\mathsf{plus}\, v\, (\mathsf{sumRangeBodyFix}\, (\mathsf{plus}\, v\, \mathsf{one})\, v')$ converges to a closed value representing $n + 0 = n = +[n : m]$. Thus $\mathsf{sumRangeBodyFix}\, v\, v'$ converges to a closed value representing $+[n : m]$.
- Suppose $n > m$. Then $+[n : m] = 0$. By Lemma 2.7, we have that $\mathsf{sumRangeBodyFix}\, v\, v' \rightarrow^* \mathsf{zero}$, and zero represents $0 = +[n : m]$.

(Inductive Step) Suppose $l \in \mathbb{N}$, and assume the inductive hypothesis, $P(l)$. We must show $P(l+1)$. Suppose v and v' are closed values, $n, m \in \mathbb{N}$, $m -_{\mathbb{N}} n = l + 1$, v represents n, and v' represents m. We must show that

$$\mathsf{sumRangeBodyFix}\, v\, v'$$

converges to a closed value representing $+[n : m]$. Since $m -_{\mathbb{N}} n = l + 1 > 0$, we have that $n < m$ and $m - n = l + 1$. Hence $n + 1 \leq m$, $+[n : m] = n + +[n + 1 : m]$, and $m -_{\mathbb{N}} (n + 1) = m - (n + 1) = m - n - 1 = l$. Since $n < m$, Lemma 2.7 tells us that $\mathsf{sumRangeBodyFix}\, v\, v' \rightarrow^* \mathsf{plus}\, v\, (\mathsf{sumRangeBodyFix}\, (\mathsf{plus}\, v\, \mathsf{one})\, v')$. By Proposition J, we have that $\mathsf{plus}\, v\, \mathsf{one} \rightarrow^* u$, for a closed value u representing $n + 1$. Since u and v' are closed values representing $n + 1$ and m, respectively, and $m -_{\mathbb{N}} (n + 1) = l$, the inductive hypothesis, $P(l)$, tells us that $\mathsf{sumRangeBodyFix}\, u\, v' \rightarrow^* u'$ for a closed value u' representing $+[n + 1 : m]$. By Proposition G, it follows that $\mathsf{sumRangeBodyFix}\, (\mathsf{plus}\, v\, \mathsf{one})\, v' \rightarrow^* u'$. Since v and u' are closed values representing n and $+[n + 1 : m]$, respectively, Proposition J tells us that $\mathsf{plus}\, v\, (\mathsf{sumRangeBodyFix}\, (\mathsf{plus}\, v\, \mathsf{one})\, v')$ converges to a closed value representing $n + +[n + 1 : m] = +[n : m]$. Thus $\mathsf{sumRangeBodyFix}\, v\, v'$ converges to a closed value representing $+[n : m]$.

□

Now, to prove the result of the exercise, suppose t and t' are closed terms, $n, m \in \mathbb{N}$, t converges to a closed value representing n, and t' converges to a closed value representing m. We must show that $\mathsf{sumRange}\, t\, t'$ converges to a closed value representing $+[n:m]$. By Lemmas 2.6(1) and 2.8, we have that

$$\begin{aligned}\mathsf{sumRange}\, \bar{t}\, \overline{t'} &\to^* \mathsf{sumRangeBodyFix}\, \bar{t}\, \overline{t'} \\ &\to^* v,\end{aligned}$$

where v is a closed value representing $+[n:m]$. By Proposition G, it follows that $\mathsf{sumRange}\, t\, t'$ converges to v.