CIS 705 — Programming Languages — Spring 2010

Assignment 6

Model Answers

Exercise 1

Lemma 1.1
For all contexts Γ, λ_E terms t and types T, if $\Gamma \vdash_E t : T$, then $\Gamma \vdash_I e(t) : T$.

Proof. Let $P(\Gamma, t, T)$ be the predicate between contexts, λ_E terms and types that is defined by: $P(\Gamma, t, T)$ iff $\Gamma \vdash_I e(t) : T$. We use induction on the typing relation of the external language to show that, for all contexts Γ, λ_E terms t and types T, if $\Gamma \vdash_E t : T$, then $P(\Gamma, t, T)$.

(T-True) Suppose Γ is a context. We must show that $P(\Gamma, \mathsf{true}, \mathsf{Bool})$. We have that $\Gamma \vdash_I \mathsf{true} : \mathsf{Bool}$. And $e(\mathsf{true}) = \mathsf{true}$, so that $\Gamma \vdash_I e(\mathsf{true}) : \mathsf{Bool}$, i.e., $P(\Gamma, \mathsf{true}, \mathsf{Bool})$.

(T-False) Suppose Γ is a context. We must show that $P(\Gamma, \mathsf{false}, \mathsf{Bool})$. We have that $\Gamma \vdash_I \mathsf{false} : \mathsf{Bool}$. And $e(\mathsf{false}) = \mathsf{false}$, so that $\Gamma \vdash_I e(\mathsf{false}) : \mathsf{Bool}$, i.e., $P(\Gamma, \mathsf{false}, \mathsf{Bool})$.

(T-If) Suppose Γ is a context, t_1, t_2 and t_3 are λ_E terms, T is a type, $\Gamma \vdash_E t_1 : \mathsf{Bool}$, $\Gamma \vdash_E t_2 : T$ and $\Gamma \vdash_E t_3 : T$, and assume the inductive hypothesis, $P(\Gamma, t_1, \mathsf{Bool})$, $P(\Gamma, t_2, T)$ and $P(\Gamma, t_3, T)$. We must show that $P(\Gamma, \mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3, T)$. By the inductive hypothesis, we have that $\Gamma \vdash_I e(t_1) : \mathsf{Bool}$, $\Gamma \vdash_I e(t_2) : T$ and $\Gamma \vdash_I e(t_3) : T$. Thus $\Gamma \vdash_I \mathsf{if}\ e(t_1)\ \mathsf{then}\ e(t_2)\ \mathsf{else}\ e(t_3) : T$. But $e(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3) = \mathsf{if}\ e(t_1)\ \mathsf{then}\ e(t_2)\ \mathsf{else}\ e(t_3)$, and thus $\Gamma \vdash_I e(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3) : T$, i.e., $P(\Gamma, \mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3, T)$.

(T-Var) Suppose Γ is a context, x is a variable, T is a type, and $(x, T) \in \Gamma$. We must show that $P(\Gamma, x, T)$. Since $(x, T) \in \Gamma$, we have that $\Gamma \vdash_I x : T$. But $e(x) = x$, and thus $\Gamma \vdash_I e(x) : T$, i.e., $P(\Gamma, x, T)$.

(T-Abs) Suppose Γ is a context, x is a variable, T_1 and T_2 are types, t is a λ_E term, and $\Gamma[x \mapsto T_1] \vdash_E t : T_2$, and assume the inductive hypothesis, $P(\Gamma[x \mapsto T_1], t, T_2)$. We must show that $P(\Gamma, \lambda x : T_1.\, t, T_1 \to T_2)$. By the inductive hypothesis, we have that $\Gamma[x \mapsto T_1] \vdash_I e(t) : T_2$, so that $\Gamma \vdash_I \lambda x : T_1.\, e(t) : T_1 \to T_2$. But $e(\lambda x : T_1.\, t) = \lambda x : T_1.\, e(t)$, and thus $\Gamma \vdash_I e(\lambda x : T_1.\, t) : T_1 \to T_2$, i.e., $P(\Gamma, \lambda x : T_1.\, t, T_1 \to T_2)$.

(T-WildcardAbs) Suppose Γ is a context, T_1 and T_2 are types, t is a λ_E term, and $\Gamma \vdash_E t : T_2$, and assume the inductive hypothesis, $P(\Gamma, t, T_2)$. We must show that $P(\Gamma, \lambda - : T_1.\, t, T_1 \to T_2)$. By the inductive hypothesis, we have that $\Gamma \vdash_I e(t) : T_2$, so that $\Gamma \vdash_I \lambda - : T_1.\, e(t) : T_1 \to T_2$. But $e(\lambda - : T_1.\, t) = \lambda - : T_1.\, e(t)$, and thus $\Gamma \vdash_I e(\lambda - : T_1.\, t) : T_1 \to T_2$, i.e., $P(\Gamma, \lambda - : T_1.\, t, T_1 \to T_2)$.

(T-App) Suppose Γ is a context, t_1 and t_2 are λ_E terms, T_1 and T_2 are types, $\Gamma \vdash_E t_1 : T_1 \to T_2$ and $\Gamma \vdash_E t_2 : T_1$, and assume the inductive hypothesis, $P(\Gamma, t_1, T_1 \to T_2)$ and $P(\Gamma, t_2, T_1)$. We must show that $P(\Gamma, t_1\, t_2, T_2)$. By the inductive hypothesis, we have that $\Gamma \vdash_I e(t_1) : T_1 \to T_2$ and $\Gamma \vdash_I e(t_2) : T_1$, so that $\Gamma \vdash_I e(t_1)\, e(t_2) : T_2$. But $e(t_1\, t_2) = e(t_1)\, e(t_2)$, and thus $\Gamma \vdash_I e(t_1\, t_2) : T_2$, i.e., $P(\Gamma, t_1\, t_2, T_2)$.

(T-Ascribe) Suppose Γ is a context, t is a λ_E term, T is a type, and $\Gamma \vdash_E t : T$, and assume the inductive hypothesis, $P(\Gamma, t, T)$. We must show that $P(\Gamma, t \,\mathsf{as}\, T, T)$, i.e., $\Gamma \vdash_I e(t \,\mathsf{as}\, T) : T$. Since $e(t \,\mathsf{as}\, T) = (\lambda x : \mathsf{Bool} \to T.\, x\, \mathsf{true})(\lambda - : \mathsf{Bool}.\, e(t))$, where x is the least variable, we must show that $\Gamma \vdash_I (\lambda x : \mathsf{Bool} \to T.\, x\, \mathsf{true})(\lambda - : \mathsf{Bool}.\, e(t)) : T$. Since $(x, \mathsf{Bool} \to T) \in \Gamma[x \mapsto \mathsf{Bool} \to T]$, we have that $\Gamma[x \mapsto \mathsf{Bool} \to T] \vdash_I x : \mathsf{Bool} \to T$. And $\Gamma[x \mapsto \mathsf{Bool} \to T] \vdash_I \mathsf{true} : \mathsf{Bool}$, so that $\Gamma[x \mapsto \mathsf{Bool} \to T] \vdash_I x\, \mathsf{true} : T$. Hence $\Gamma \vdash_I \lambda x : \mathsf{Bool} \to T.\, x\, \mathsf{true} : (\mathsf{Bool} \to T) \to T$. By the inductive hypothesis, we have that $\Gamma \vdash_I e(t) : T$. Thus $\Gamma \vdash_I \lambda - : \mathsf{Bool}.\, e(t) : \mathsf{Bool} \to T$. Finally, since $\Gamma \vdash_I \lambda x : \mathsf{Bool} \to T.\, x\, \mathsf{true} : (\mathsf{Bool} \to T) \to T$ and $\Gamma \vdash_I \lambda - : \mathsf{Bool}.\, e(t) : \mathsf{Bool} \to T$, it follows that $\Gamma \vdash_I (\lambda x : \mathsf{Bool} \to T.\, x\, \mathsf{true})(\lambda - : \mathsf{Bool}.\, e(t)) : T$.

□

Lemma 1.2
For all contexts Γ, λ_E terms t, and types T, if $\Gamma \vdash_I e(t) : T$, then $\Gamma \vdash_E t : T$.

Proof. Let $P(t)$ be the predicate on λ_E terms defined by: $P(t)$ iff, for all contexts Γ and types T, if $\Gamma \vdash_I e(t) : T$, then $\Gamma \vdash_E t : T$. We use structural induction on λ_E terms to show that, for all λ_E terms t, $P(t)$.

(Constant True) We must show that $P(\mathsf{true})$. Suppose Γ is a context, T is a type, and $\Gamma \vdash_I e(\mathsf{true}) : T$. We must show that $\Gamma \vdash_E \mathsf{true} : T$. Since $e(\mathsf{true}) = \mathsf{true}$, we have that $\Gamma \vdash_I \mathsf{true} : T$. Thus, by the inversion lemma for the typing relation of λ_I, we have that $T = \mathsf{Bool}$. Hence $\Gamma \vdash_E \mathsf{true} : \mathsf{Bool} = T$.

(Constant False) We must show that $P(\mathsf{false})$. Suppose Γ is a context, T is a type, and $\Gamma \vdash_I e(\mathsf{false}) : T$. We must show that $\Gamma \vdash_E \mathsf{false} : T$. Since $e(\mathsf{false}) = \mathsf{false}$, we have that $\Gamma \vdash_I \mathsf{false} : T$. Thus, by the inversion lemma for the typing relation of λ_I, we have that $T = \mathsf{Bool}$. Hence $\Gamma \vdash_E \mathsf{false} : \mathsf{Bool} = T$.

(Conditional) Suppose t_1, t_2 and t_3 are λ_E terms, and assume the inductive hypothesis, $P(t_1)$, $P(t_2)$ and $P(t_3)$. We must show that $P(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3)$. Suppose Γ is a context, T is a type, and $\Gamma \vdash_I e(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3) : T$. We must show that $\Gamma \vdash_E \mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 : T$. Since $e(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3) = \mathsf{if}\ e(t_1)\ \mathsf{then}\ e(t_2)\ \mathsf{else}\ e(t_3)$, we have that $\Gamma \vdash_I \mathsf{if}\ e(t_1)\ \mathsf{then}\ e(t_2)\ \mathsf{else}\ e(t_3) : T$. By the inversion lemma for the typing relation of λ_I, we have that $\Gamma \vdash_I e(t_1) : \mathsf{Bool}$, $\Gamma \vdash_I e(t_2) : T$ and $\Gamma \vdash_I e(t_3) : T$. Thus, by the inductive hypothesis, it follows that $\Gamma \vdash_E t_1 : \mathsf{Bool}$, $\Gamma \vdash_E t_2 : T$ and $\Gamma \vdash_E t_3 : T$. Hence $\Gamma \vdash_E \mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 : T$.

(Variable) Suppose x is a variable. We must show that $P(x)$. Suppose Γ is a context, T is a type, and $\Gamma \vdash_I e(x) : T$. We must show that $\Gamma \vdash_E x : T$. Since $e(x) = x$, we have that $\Gamma \vdash_I x : T$. By the inversion lemma for the typing relation of λ_I, we have that $(x, T) \in \Gamma$. Thus $\Gamma \vdash_E x : T$.

(Abstraction) Suppose x is a variable, T is a type, and t is a λ_E term, and suppose the inductive hypothesis, $P(t)$. We must show that $P(\lambda x : T.\, t)$. Suppose Γ is a context, U is a type, and $\Gamma \vdash_I e(\lambda x : T.\, t) : U$. We must show that $\Gamma \vdash_E \lambda x : T.\, t : U$. Since $e(\lambda x : T.\, t) = \lambda x : T.\, e(t)$, we have that $\Gamma \vdash_I \lambda x : T.\, e(t) : U$. By the inversion lemma for the typing relation of λ_I, it follows that there is a type U' such that $U = T \to U'$ and $\Gamma[x \mapsto T] \vdash_I e(t) : U'$. Thus, by the inductive hypothesis, it follows that $\Gamma[x \mapsto T] \vdash_E t : U'$, so that $\Gamma \vdash_E \lambda x : T.\, t : T \to U' = U$.

(Wildcard Abstraction) Suppose T is a type and t is a λ_E term, and assume the inductive hypothesis, $P(t)$. We must show that $P(\lambda - : T.\, t)$. Suppose Γ is a context, U is a type, and $\Gamma \vdash_I e(\lambda - : T.\, t) : U$. We must show that $\Gamma \vdash_E \lambda - : T.\, t : U$. Since $e(\lambda - : T.\, t) = \lambda - : T.\, e(t)$, we have that $\Gamma \vdash_I \lambda - : T.\, e(t) : U$. By the inversion lemma for the typing relation of λ_I, it follows that there is a type U' such that $U = T \to U'$ and $\Gamma \vdash_I e(t) : U'$. Thus, by the inductive hypothesis, it follows that $\Gamma \vdash_E t : U'$, so that $\Gamma \vdash_E \lambda - : T.\, t : T \to U' = U$.

(Application) Suppose t_1 and t_2 are λ_E terms, and assume the inductive hypothesis, $P(t_1)$ and $P(t_2)$. We must show that $P(t_1\, t_2)$. Suppose Γ is a context, T is a type, and $\Gamma \vdash_I e(t_1\, t_2) : T$. We must show that $\Gamma \vdash_E t_1\, t_2 : T$. Since $e(t_1\, t_2) = e(t_1)\, e(t_2)$, we have that $\Gamma \vdash_I e(t_1)\, e(t_2) : T$. By the inversion lemma for the typing relation of λ_I, we have that there is a type T' such that $\Gamma \vdash_I e(t_1) : T' \to T$ and $\Gamma \vdash_I e(t_2) : T'$. Thus, by the inductive hypothesis, it follows that $\Gamma \vdash_E t_1 : T' \to T$ and $\Gamma \vdash_E t_2 : T'$, so that $\Gamma \vdash_E t_1\, t_2 : T$.

(Ascription) Suppose t is a λ_E term and T is a type, and assume the inductive hypothesis, $P(t)$. We must show that $P(t \;\mathsf{as}\; T)$. Suppose Γ is a context, U is a type, and $\Gamma \vdash_I e(t \;\mathsf{as}\; T) : U$. We must show that $\Gamma \vdash_E t \;\mathsf{as}\; T : U$. Since $e(t \;\mathsf{as}\; T) = (\lambda x : \mathsf{Bool} \to T.\, x\, \mathsf{true})(\lambda - : \mathsf{Bool}.\, e(t))$, where x is the least variable, we have that $\Gamma \vdash_I (\lambda x : \mathsf{Bool} \to T.\, x\, \mathsf{true})(\lambda - : \mathsf{Bool}.\, e(t)) : U$. By the inversion lemma for the typing relation of λ_I, we have that there is a type U' such that $\Gamma \vdash_I \lambda x : \mathsf{Bool} \to T.\, x\, \mathsf{true} : U' \to U$ and $\Gamma \vdash_I \lambda - : \mathsf{Bool}.\, e(t) : U'$. Since $(x, \mathsf{Bool} \to T) \in \Gamma[x \mapsto \mathsf{Bool} \to T]$, we have that $\Gamma[x \mapsto \mathsf{Bool} \to T] \vdash_I x : \mathsf{Bool} \to T$. And $\Gamma[x \mapsto \mathsf{Bool} \to T] \vdash_I \mathsf{true} : \mathsf{Bool}$, so that $\Gamma[x \mapsto \mathsf{Bool} \to T] \vdash_I x\, \mathsf{true} : T$. Hence $\Gamma \vdash_I \lambda x : \mathsf{Bool} \to T.\, x\, \mathsf{true} : (\mathsf{Bool} \to T) \to T$. Thus, by the Uniqueness of Typing Proposition for λ_I, it follows that $(\mathsf{Bool} \to T) \to T = U' \to U$, so that $U' = \mathsf{Bool} \to T$ and $U = T$. Thus it will suffice to show that $\Gamma \vdash_E t \;\mathsf{as}\; T : T$. Since $\Gamma \vdash_I \lambda - : \mathsf{Bool}.\, e(t) : U' = \mathsf{Bool} \to T$, the inversion lemma for the typing relation on λ_I tells us that $\Gamma \vdash_I e(t) : T$. Thus, by the inductive hypothesis, we have that $\Gamma \vdash_E t : T$, so that $\Gamma \vdash_E t \;\mathsf{as}\; T : T$.

□

Suppose t is a λ_E term. We must show that

$$\emptyset \vdash_E t : \mathsf{Bool} \quad \text{iff} \quad \emptyset \vdash_I e(t) : \mathsf{Bool}.$$

The are two directions to show.

- Suppose $\emptyset \vdash_E t : \mathsf{Bool}$. By Lemma 1.1, we have that $\emptyset \vdash_I e(t) : \mathsf{Bool}$.
- Suppose $\emptyset \vdash_I e(t) : \mathsf{Bool}$. By Lemma 1.2, we have that $\emptyset \vdash_E t : \mathsf{Bool}$.

Exercise 2

Lemma 2.1

(1) *For all closed λ_I terms t_1, t_1', t_2 and t_3, if $t_1 \to_I^* t_1'$, then* $\mathsf{if}\; t_1 \;\mathsf{then}\; t_2 \;\mathsf{else}\; t_3 \to_I^* \mathsf{if}\; t_1' \;\mathsf{then}\; t_2 \;\mathsf{else}\; t_3$.

(2) For all closed λ_I terms t_1, t'_1 and t_2, if $t_1 \rightarrow_I^* t'_1$, then $t_1\, t_2 \rightarrow_I^* t'_1\, t_2$.

(3) For all closed λ_I values v_1, and closed λ_I terms t_2 and t'_2, if $t_2 \rightarrow_I^* t'_2$, then $v_1\, t_2 \rightarrow_I^* v_1\, t'_2$.

Proof. Simple inductions on $\rightarrow_I^*$. $\square$

Lemma 2.2
For all closed λ_E terms t and t', if $t \rightarrow_E t'$, then $e(t) \rightarrow_I^* e(t')$.

Proof. Let $P(t, t')$ be the predicate on closed λ_E terms defined by: $P(t, t')$ iff $e(t) \rightarrow_I^* e(t')$. We use induction on the evaluation relation of the external language to show that, for all closed λ_E terms t and t', if $t \rightarrow_E t'$, then $P(t, t')$.

(E-IfTrue) Suppose t_2 and t_3 are closed λ_E terms. We must show that $P(\textsf{if true then } t_2 \textsf{ else } t_3, t_2)$. We have that $e(\textsf{if true then } t_2 \textsf{ else } t_3) = \textsf{if } e(\textsf{true}) \textsf{ then } e(t_2) \textsf{ else } e(t_3) = \textsf{if true then } e(t_2) \textsf{ else } e(t_3) \rightarrow_I e(t_2)$, showing that $e(\textsf{if true then } t_2 \textsf{ else } t_3) \rightarrow_I^* e(t_2)$.

(E-IfFalse) Suppose t_2 and t_3 are closed λ_E terms. We must show that $P(\textsf{if false then } t_2 \textsf{ else } t_3, t_3)$. We have that $e(\textsf{if false then } t_2 \textsf{ else } t_3) = \textsf{if } e(\textsf{false}) \textsf{ then } e(t_2) \textsf{ else } e(t_3) = \textsf{if false then } e(t_2) \textsf{ else } e(t_3) \rightarrow_I e(t_3)$, showing that $e(\textsf{if false then } t_2 \textsf{ else } t_3) \rightarrow_I^* e(t_3)$.

(E-If) Suppose t_1, t'_1, t_2 and t_3 are closed λ_E terms, and $t_1 \rightarrow_E t'_1$, and assume the inductive hypothesis, $P(t_1, t'_1)$, i.e., $e(t_1) \rightarrow_I^* e(t'_1)$. We must show that $P(\textsf{if } t_1 \textsf{ then } t_2 \textsf{ else } t_3, \textsf{if } t'_1 \textsf{ then } t_2 \textsf{ else } t_3)$. We have that $e(\textsf{if } t_1 \textsf{ then } t_2 \textsf{ else } t_3) = \textsf{if } e(t_1) \textsf{ then } e(t_2) \textsf{ else } e(t_3) \rightarrow_I^* \textsf{if } e(t'_1) \textsf{ then } e(t_2) \textsf{ else } e(t_3) = e(\textsf{if } t'_1 \textsf{ then } t_2 \textsf{ else } t_3)$, by Lemma 2.1(1), showing that $e(\textsf{if } t_1 \textsf{ then } t_2 \textsf{ else } t_3) \rightarrow_I^* e(\textsf{if } t'_1 \textsf{ then } t_2 \textsf{ else } t_3)$.

(E-App1) Suppose t_1, t'_1 and t_2 are closed λ_E terms, and $t_1 \rightarrow_E t'_1$, and assume the inductive hypothesis, $P(t_1, t'_1)$, i.e., $e(t_1) \rightarrow_I^* e(t'_1)$. We must show that $P(t_1\, t_2, t'_1\, t_2)$. We have that $e(t_1\, t_2) = e(t_1)\, e(t_2) \rightarrow_I^* e(t'_1)\, e(t_2) = e(t'_1\, t_2)$, by Lemma 2.1(2), showing that $e(t_1\, t_2) \rightarrow_I^* e(t'_1\, t_2)$.

(E-App2) Suppose v_1 is a closed λ_E value, t_2 and t'_2 are closed λ_E terms, and $t_2 \rightarrow_E t'_2$, and assume the inductive hypothesis, $P(t_2, t'_2)$, i.e., $e(t_2) \rightarrow_I^* e(t'_2)$. We must show that $P(v_1\, t_2, v_1\, t'_2)$. Since v_1 is a closed λ_E value, we have that $e(v_1)$ is a closed λ_I value. Thus $e(v_1\, t_2) = e(v_1)\, e(t_2) \rightarrow_I^* e(v_1)\, e(t'_2) = e(v_1\, t'_2)$, by Lemma 2.1(3), showing that $e(v_1\, t_2) \rightarrow_I^* e(v_1\, t'_2)$.

(E-AppAbs) Suppose x is a variable, T is a type, t is a λ_E term such that $\mathsf{FV}(t) \subseteq \{x\}$, and v is a closed λ_E value. We must show that $P((\lambda x : T.\, t)v, [x \mapsto v]t)$. We have that $\mathsf{FV}(e(t)) \subseteq \{x\}$. Since v is a closed λ_E value, we have that $e(v)$ is a closed λ_I value. Thus $e((\lambda x : T.\, t)v) = e(\lambda x : T.\, t)\, e(v) = (\lambda x : T.\, e(t))\, e(v) \rightarrow_I [x \mapsto e(v)](e(t)) = e([x \mapsto v]t)$, by the Elaboration of Substitutions Proposition, showing that $e((\lambda x : T.\, t)v) \rightarrow_I^* e([x \mapsto v]t)$.

(E-AppWildcardAbs) Suppose T is a type, t' is a closed λ_E term, and v is a closed λ_E value. We must show that $P((\lambda - : T.\, t')v, t')$. Since v is a closed λ_E value, we have that $e(v)$ is a closed λ_I value. Thus $e((\lambda - : T.\, t')v) = e(\lambda - : T.\, t')\, e(v) = (\lambda - : T.\, e(t'))\, e(v) \rightarrow_I e(t')$, showing that $e((\lambda - : T.\, t')v) \rightarrow_I^* e(t')$.

(E-AscribeEager) Suppose t is a closed λ_E term and T is a type. We must show that $P(t \textsf{ as } T, t)$. We have that $e(t \textsf{ as } T) = (\lambda x : \textsf{Bool} \rightarrow T.\, x\, \textsf{true})(\lambda - : \textsf{Bool}.\, e(t))$, where x is the least variable. Thus $e(t \textsf{ as } T) \rightarrow_I (\lambda - : \textsf{Bool}.\, e(t))\textsf{true} \rightarrow_I e(t)$, showing that $e(t \textsf{ as } T) \rightarrow_I^* e(t)$.

□

Lemma 2.3
For all λ_E terms t, for all closed λ_I terms u, if t is closed and $e(t) \to_I u$, then there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $t \to_E t'$.

Proof. Let $P(t)$ be the predicate on λ_E terms defined by: $P(t)$ iff, for all closed λ_I terms u, if t is closed and $e(t) \to_I u$, then there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $t \to_E t'$. We use structural induction on λ_E terms to show that, for all λ_E terms t, $P(t)$.

- **(Constant True)** We must show that $P(\mathsf{true})$. Suppose u is a closed λ_I term, true is closed, and $e(\mathsf{true}) \to_I u$. We must show that there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $\mathsf{true} \to_E t'$. But $e(\mathsf{true}) = \mathsf{true}$, and thus $\mathsf{true} \to_I u$—contradiction. Thus there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $\mathsf{true} \to_E t'$.

- **(Constant False)** We must show that $P(\mathsf{false})$. Suppose u is a closed λ_I term, false is closed, and $e(\mathsf{false}) \to_I u$. We must show that there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $\mathsf{false} \to_E t'$. But $e(\mathsf{false}) = \mathsf{false}$, and thus $\mathsf{false} \to_I u$—contradiction. Thus there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $\mathsf{false} \to_E t'$.

- **(Conditional)** Suppose t_1, t_2 and t_3 are λ_E terms, and assume the inductive hypothesis, $P(t_1)$, $P(t_2)$ and $P(t_3)$. We must show that $P(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3)$. Suppose u is a closed λ_I term, $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$ is closed, and $e(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3) \to_I u$. We must show that there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to_E t'$. Since $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$ is closed, we have that t_1, t_2 and t_3 are closed, so that $e(t_1)$, $e(t_2)$ and $e(t_3)$ are also closed. Since $e(\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3) = \mathsf{if}\ e(t_1)\ \mathsf{then}\ e(t_2)\ \mathsf{else}\ e(t_3)$, we have that $\mathsf{if}\ e(t_1)\ \mathsf{then}\ e(t_2)\ \mathsf{else}\ e(t_3) \to_I u$. By the inversion lemma for the evaluation relation of λ_I, there are three cases to consider.

  - **(E-IfTrue)** Suppose $e(t_1) = \mathsf{true}$ and $u = e(t_2)$. By the definition of the elaboration function, we have that $t_1 = \mathsf{true}$. Thus $u \to_I^* e(t_2)$ and $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 = \mathsf{if}\ \mathsf{true}\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to_E t_2$, showing that $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to_E t_2$.

  - **(E-IfFalse)** Suppose $e(t_1) = \mathsf{false}$ and $u = e(t_3)$. By the definition of the elaboration function, we have that $t_1 = \mathsf{false}$. Thus $u \to_I^* e(t_3)$ and $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 = \mathsf{if}\ \mathsf{false}\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to_E t_3$, showing that $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to_E t_3$.

  - **(E-If)** Suppose there is a closed λ_I term u' such that $e(t_1) \to_I u'$ and $u = \mathsf{if}\ u'\ \mathsf{then}\ e(t_2)\ \mathsf{else}\ e(t_3)$. By $P(t_1)$, we have that there is a closed λ_E term t_1' such that $u' \to_I^* e(t_1')$ and $t_1 \to_E t_1'$. Thus $u = \mathsf{if}\ u'\ \mathsf{then}\ e(t_2)\ \mathsf{else}\ e(t_3) \to_I^* \mathsf{if}\ e(t_1')\ \mathsf{then}\ e(t_2)\ \mathsf{else}\ e(t_3) = e(\mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3)$, by Lemma 2.1(1), showing that $u \to_I^* e(\mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3)$. And $\mathsf{if}\ t_1\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3 \to_E \mathsf{if}\ t_1'\ \mathsf{then}\ t_2\ \mathsf{else}\ t_3$.

- **(Variable)** Suppose x is a variable. We must show that $P(x)$. Suppose u is a closed λ_I term, x is closed, and $e(x) \to_I u$. We must show that there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $x \to_E t'$. But x is not closed—contradiction. Thus there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $x \to_E t'$.

(Abstraction) Suppose x is a variable, T is a type, and t is a λ_E term, and suppose the inductive hypothesis, $P(t)$. We must show that $P(\lambda x : T.\, t)$. Suppose u is a closed λ_I term, $\lambda x : T.\, t$ is closed, and $e(\lambda x : T.\, t) \to_I u$. We must show that there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $\lambda x : T.\, t \to_E t'$. Since $e(\lambda x : T.\, t) = \lambda x : T.\, e(t)$, we have that $\lambda x : T.\, e(t) \to_I u$— contradiction. Thus there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $\lambda x : T.\, t \to_E t'$.

(Wildcard Abstraction) Suppose T is a type and t is a λ_E term, and assume the inductive hypothesis, $P(t)$. We must show that $P(\lambda - : T.\, t)$. Suppose u is a closed λ_I term, $\lambda - : T.\, t$ is closed, and $e(\lambda - : T.\, t) \to_I u$. We must show that there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $\lambda - : T.\, t \to_E t'$. Since $e(\lambda - : T.\, t) = \lambda - : T.\, e(t)$, we have that $\lambda - : T.\, e(t) \to_I u$— contradiction. Thus there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $\lambda - : T.\, t \to_E t'$.

(Application) Suppose t_1 and t_2 are λ_E terms, and assume the inductive hypothesis, $P(t_1)$ and $P(t_2)$. We must show that $P(t_1\, t_2)$. Suppose u is a closed λ_I term, $t_1\, t_2$ is closed, and $e(t_1\, t_2) \to_I u$. We must show that there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $t_1\, t_2 \to_E t'$. Since $t_1\, t_2$ is closed, we have that t_1 and t_2 are closed, so that $e(t_1)$ and $e(t_2)$ are also closed. Because $e(t_1\, t_2) = e(t_1)\, e(t_2)$, we have that $e(t_1)\, e(t_2) \to_I u$. By the inversion lemma for the evaluation relation of λ_I, there are four cases to consider.

(E-App1) Suppose there is a closed λ_I term u' such that $e(t_1) \to_I u'$ and $u = u'\, e(t_2)$. By $P(t_1)$, we have that there is a closed λ_E term t_1' such that $u' \to_I^* e(t_1')$ and $t_1 \to_E t_1'$. Thus $u = u'\, e(t_2) \to_I^* e(t_1')\, e(t_2) = e(t_1'\, t_2)$, by Lemma 2.1(2), showing that $u \to_I^* e(t_1'\, t_2)$. And $t_1\, t_2 \to_E t_1'\, t_2$.

(E-App2) Suppose $e(t_1)$ is a λ_I value, and there is a closed λ_I term u' such that $e(t_2) \to_I u'$ and $u = e(t_1)\, u'$. By $P(t_2)$, we have that there is a closed λ_E term t_2' such that $u' \to_I^* e(t_2')$ and $t_2 \to_E t_2'$. Thus $u = e(t_1)\, u' \to_I^* e(t_1)\, e(t_2') = e(t_1\, t_2')$, by Lemma 2.1(3), showing that $u \to_I^* e(t_1\, t_2')$. Since $e(t_1)$ is a λ_I value, it follows that t_1 is a λ_E value. Hence $t_1\, t_2 \to_E t_1\, t_2'$.

(E-AppAbs) Suppose $e(t_2)$ is a λ_I value, and there is a variable x, a type T and a λ_I term u' with the property that $\mathsf{FV}(u') \subseteq \{x\}$ such that $e(t_1) = \lambda x : T.\, u'$ and $u = [x \mapsto e(t_2)]u'$. By the definition of the elaboration function, it follows that $t_1 = \lambda x : T.\, t'$ for some λ_E term t' such that $e(t') = u'$, so that $\mathsf{FV}(t') \subseteq \{x\}$. Thus $u = [x \mapsto e(t_2)]u' = [x \mapsto e(t_2)](e(t')) = e([x \mapsto t_2]t')$, by the Elaboration of Substitution Proposition, showing that $u \to_I^* e([x \mapsto t_2]t')$. Since $e(t_2)$ is a λ_I value, it follows that t_2 is a λ_E value. Thus $t_1\, t_2 = (\lambda x : T.\, t')t_2 \to_E [x \mapsto t_2]t'$.

(E-AppWildcardAbs) Suppose $e(t_2)$ is a λ_I value, and there is a type T and a closed λ_I term u' such that $e(t_1) = \lambda - : T.\, u'$ and $u = u'$. By the definition of the elaboration function, it follows that $t_1 = \lambda - : T.\, t'$ for some λ_E term t' such that $e(t') = u'$, so that t' is also closed. Thus $u = u' = e(t')$, showing that $u \to_I^* e(t')$. Since $e(t_2)$ is a λ_I value, it follows that t_2 is a λ_E value. Thus $t_1\, t_2 = (\lambda - : T.\, t')t_2 \to_E t'$.

(Ascription) Suppose t is a λ_E term and T is a type, and assume the inductive hypothesis, $P(t)$. We must show that $P(t \;\mathsf{as}\; T)$. Suppose u is a closed λ_I term, $t \;\mathsf{as}\; T$ is closed and $e(t \;\mathsf{as}\; T) \to_I u$. We must show that there is a closed λ_E term t' such that $u \to_I^* e(t')$ and $t \;\mathsf{as}\; T \to_E t'$. Since $t \;\mathsf{as}\; T$ is closed, we have that t is closed. Since $e(t \;\mathsf{as}\; T) = (\lambda x : \mathsf{Bool} \to T.\, x\, \mathsf{true})(\lambda - : \mathsf{Bool}.\, e(t))$, where x is the least variable, we have that $(\lambda x : \mathsf{Bool} \to T.\, x\, \mathsf{true})(\lambda - : \mathsf{Bool}.\, e(t)) \to_I u$, and thus that $u = (\lambda - : \mathsf{Bool}.\, e(t))\mathsf{true}$. Thus $u \to_I e(t)$, showing that $u \to_I^* e(t)$. And $t \;\mathsf{as}\; T \to_E t$.

□

Lemma 2.4
For all closed λ_E terms t and t', if $t \rightarrow_E^* t'$, then $e(t) \rightarrow_I^* e(t')$.

Proof. Define a predicate $P(t, t')$ on closed λ_E terms by: $P(t, t')$ iff $e(t) \rightarrow_I^* e(t')$. We use induction on $\rightarrow_E^*$ to show that, for all closed λ_E terms t and t', if $t \rightarrow_E^* t'$, then $P(t, t')$.

(Eval) Suppose t and t' are closed λ_E terms and $t \rightarrow_E t'$. We must show that $P(t, t')$. By Lemma 2.2, we have that $e(t) \rightarrow_I^* e(t')$, i.e., $P(t, t')$.

(Refl) Suppose t is a closed λ_E term. Then $e(t) \rightarrow_I^* e(t)$, i.e., $P(t, t)$.

(Trans) Suppose t_1, t_2 and t_3 are closed λ_E terms, $t_1 \rightarrow_E^* t_2$ and $t_2 \rightarrow_E^* t_3$, and assume the inductive hypothesis, $P(t_1, t_2)$ and $P(t_2, t_3)$. We must show that $P(t_1, t_3)$. By the inductive hypothesis, we have that $e(t_1) \rightarrow_I^* e(t_2) \rightarrow_I^* e(t_3)$. Thus $e(t_1) \rightarrow_I^* e(t_3)$, i.e., $P(t_1, t_3)$.

□

Lemma 2.5
For all closed λ_I terms u and u', if $u \rightarrow_I^* u'$, then, for all closed λ_E terms t, if $u \rightarrow_I^* e(t)$, then there is a closed λ_E term t' such that $u' \rightarrow_I^* e(t')$ and $t \rightarrow_E^* t'$.

Proof. Define a predicate $P(u, u')$ on closed λ_I terms by: $P(u, u')$ iff, for all closed λ_E terms t, if $u \rightarrow_I^* e(t)$, then there is a closed λ_E term t' such that $u' \rightarrow_I^* e(t')$ and $t \rightarrow_E^* t'$. We use induction on $\rightarrow_I^*$ to show that, for all closed λ_I terms u and u', if $u \rightarrow_I^* u'$, then $P(u, u')$.

(Eval) Suppose u and u' are closed λ_I terms and $u \rightarrow_I u'$. We must show that $P(u, u')$. Suppose t is a closed λ_E term and $u \rightarrow_I^* e(t)$. We must show that there is a closed λ_E term t' such that $u' \rightarrow_I^* e(t')$ and $t \rightarrow_E^* t'$. Since $u \rightarrow_I^* e(t)$, we have that $u \rightarrow_I^n e(t)$ for some $n \in \mathbb{N}$. There are two cases to consider.

- Suppose $n = 0$. Thus $u \rightarrow_I^0 e(t)$, so that $u = e(t)$. Then $e(t) = u \rightarrow_I u'$, and thus Lemma 2.3 tells us that there is a closed λ_E term t' such that $u' \rightarrow_I^* e(t')$ and $t \rightarrow_E t'$. Hence $t \rightarrow_E^* t'$.
- Suppose $n \geq 1$. Thus there is a closed λ_I term u'' such that $u \rightarrow_I u'' \rightarrow_I^* e(t)$. By the Determinacy of Evaluation Proposition for λ_I, since $u \rightarrow_I u'$ and $u \rightarrow_I u''$, we have that $u' = u''$. Thus $u' \rightarrow_I^* e(t)$ and $t \rightarrow_E^* t$.

(Refl) Suppose u is a closed λ_I term. We must show that $P(u, u)$. Suppose t is a closed λ_E term, and $u \rightarrow_I^* e(t)$. We must show that there is a closed λ_E term t' such that $u \rightarrow_I^* e(t')$ and $t \rightarrow_E^* t'$. We have that $u \rightarrow_I^* e(t)$ and $t \rightarrow_E^* t$.

(Trans) Suppose u_1, u_2 and u_3 are closed λ_I terms, $u_1 \rightarrow_I^* u_2$ and $u_2 \rightarrow_I^* u_3$, and assume the inductive hypothesis, $P(u_1, u_2)$ and $P(u_2, u_3)$. We must show that $P(u_1, u_3)$. Suppose t_1 is a closed λ_E term, and $u_1 \rightarrow_I^* e(t_1)$. We must show that there is a closed λ_E term t_3 such that $u_3 \rightarrow_I^* e(t_3)$ and $t_1 \rightarrow_E^* t_3$. Since $P(u_1, u_2)$ and $u_1 \rightarrow_I^* e(t_1)$, there is a closed λ_E term t_2 such that $u_2 \rightarrow_I^* e(t_2)$ and $t_1 \rightarrow_E^* t_2$. Since $P(u_2, u_3)$ and $u_2 \rightarrow_I^* e(t_2)$, there is a closed λ_E term t_3 such that $u_3 \rightarrow_I^* e(t_3)$ and $t_2 \rightarrow_E^* t_3$. Since $t_1 \rightarrow_E^* t_2 \rightarrow_E^* t_3$, we have that $t_1 \rightarrow_E^* t_3$. Thus $u_3 \rightarrow_I^* e(t_3)$ and $t_1 \rightarrow_E^* t_3$.

□

Suppose t is a closed λ_E term and $b \in \{\mathsf{true}, \mathsf{false}\}$. We must show that

$$t \rightarrow_E^* b \quad \text{iff} \quad e(t) \rightarrow_I^* b.$$

There are two directions to show.

- Suppose $t \rightarrow_E^* b$. By Lemma 2.4, we have that $e(t) \rightarrow_I^* e(b)$. Since $e(\mathsf{true}) = \mathsf{true}$ and $e(\mathsf{false}) = \mathsf{false}$, we have that $e(b) = b$. Thus $e(t) \rightarrow_I^* e(b) = b$.
- Suppose $e(t) \rightarrow_I^* b$. By Lemma 2.5, since t is closed and $e(t) \rightarrow_I^* e(t)$, we have that there is a closed λ_E term t' such that $b \rightarrow_I^* e(t')$ and $t \rightarrow_E^* t'$. Since $b \rightarrow_I^* e(t')$, there is an $n \in \mathbb{N}$ such that $b \rightarrow_I^n e(t')$. If $n \geq 1$, then b is not a normal form, which is impossible. Thus $n = 0$, and $b \rightarrow_I^0 e(t')$, i.e., $b = e(t')$. By the definition of the elaboration function, we can see that, if $b = \mathsf{true}$, then $t' = \mathsf{true}$, and, if $b = \mathsf{false}$, then $t' = \mathsf{false}$. Thus $t' = b$, so that $t \rightarrow_E^* t' = b$.