CIS 705 — Programming Languages — Spring 2009

Assignment 4

Model Answers to Exercise 1

Exercise 1

Note that, if v is a closed value, then $\overline{v} = v$. Thus, if t is a closed term that converges, then $\overline{\overline{t}} = \overline{t}$.

Proposition D

Lemma 1.1
For all closed terms t and t', closed values v and $n, m \in \mathbb{N}$, if $t \rightarrow^n v$ and $t \rightarrow^m t'$, then $m \leq n$ and $t' \rightarrow^{n-m} v$.

Proof. Suppose, toward a contradiction, that $m > n$. Thus $m - n > 0$ and $n + (m - n) = m$. Hence, because $t \rightarrow^m t'$, we have that $t \rightarrow^{n+(m-n)} t'$, so that there is a closed term u such that $t \rightarrow^n u \rightarrow^{m-n} t'$. Since $t \rightarrow^n v$ and $t \rightarrow^n u$, we have that $v = u \rightarrow^{m-n} t'$. But $m - n > 0$, so that there is a u' such that $v \rightarrow u'$, contradicting the fact that v is a value. Thus $m \leq n$.

Since $m \leq n$, it follows that $n - m \geq 0$ and $n = m + (n - m)$. Because $t \rightarrow^n v$, we have that $t \rightarrow^{m+(n-m)} v$, so that there is a closed term u such that $t \rightarrow^m u \rightarrow^{n-m} v$. But $t \rightarrow^m t'$, so that $t' = u \rightarrow^{n-m} v$. □

Suppose t_1 and t_2 are closed terms and $t_1\, t_2$ converges. We must show that t_1, t_2 and $\overline{t_1}\ \overline{t_2}$ converge and $\overline{t_1 t_2} = \overline{\overline{t_1}\ \overline{t_2}}$. Because $t_1\, t_2$ converges and is closed, there is a closed value v such that $t_1\, t_2 \rightarrow^* v$. Thus $\overline{t_1\, t_2} = v$, and there is an $n \in \mathbb{N}$ such that $t_1\, t_2 \rightarrow^n v$.

Suppose, toward a contraction, that t_1 diverges. Thus, by Proposition C, there is a closed term t_1' such that $t_1 \rightarrow^{n+1} t_1'$. Thus $t_1\, t_2 \rightarrow^{n+1} t_1'\, t_2$, by Proposition A(1). But $t_1\, t_2 \rightarrow^n v$, so that $n+1 \leq n$, by Lemma 1.1—contradiction. Thus t_1 converges, so that there is a closed value v_1 such that $t_1 \rightarrow^* v_1$ and $\overline{t_1} = v_1$. Thus $t_1 \rightarrow^{n_1} v_1$ for some $n_1 \in \mathbb{N}$, so that $t_1\, t_2 \rightarrow^{n_1} v_1\, t_2$, by Proposition A(1). But $t_1\, t_2 \rightarrow^n v$, so that $n_1 \leq n$ and $v_1\, t_2 \rightarrow^{n-n_1} v$, by Lemma 1.1.

Suppose, toward a contradiction, that t_2 diverges. Thus, by Proposition C, there is a closed term t_2' such that $t_2 \rightarrow^{n-n_1+1} t_2'$, so that $v_1\, t_2 \rightarrow^{n-n_1+1} v_1\, t_2'$, by Proposition A(2). But $v_1\, t_2 \rightarrow^{n-n_1} v$, so that $n - n_1 + 1 \leq n - n_1$, by Lemma 1.1—contradiction. Thus t_2 converges, so that there is a closed value v_2 such that $t_2 \rightarrow^* v_2$ and $\overline{t_2} = v_2$. Hence $t_2 \rightarrow^{n_2} v_2$ for some $n_2 \in \mathbb{N}$, so that $v_1\, t_2 \rightarrow^{n_2} v_1\, v_2$, by Proposition A(2). But $v_1\, t_2 \rightarrow^{n-n_1} v$, so that $n_2 \leq n - n_1$ and $v_1\, v_2 \rightarrow^{n-n_1-n_2} v$, by Lemma 1.1. Thus $v_1\, v_2 \rightarrow^* v$.

We showed above that t_1 and t_2 converge, $\overline{t_1} = v_1$ and $\overline{t_2} = v_2$. And $\overline{t_1}\ \overline{t_2} = v_1\, v_2 \rightarrow^* v$, showing that $\overline{t_1}\ \overline{t_2}$ converges and $\overline{\overline{t_1}\ \overline{t_2}} = v$. Finally, we have that $\overline{t_1\, t_2} = v = \overline{\overline{t_1}\ \overline{t_2}}$.

Proposition E

Suppose t_1 and t_2 are closed, and t_1, t_2 and $\overline{t_1}\ \overline{t_2}$ converge. We must show that $t_1\, t_2$ converges and $\overline{t_1 t_2} = \overline{\overline{t_1}\ \overline{t_2}}$. Because t_1 converges, there is a closed value v_1 such that $t_1 \rightarrow^* v_1$, so that $\overline{t_1} = v_1$. Because t_2 converges, there is a closed value v_2 such that $t_2 \rightarrow^* v_2$, so that $\overline{t_2} = v_2$. Thus,

$v_1\, v_2 = \overline{t_1}\, \overline{t_2}$ converges, so that there is a closed value v such that $v_1\, v_2 \to^* v$. By Proposition B(1), because $t_1 \to^* v_1$, we have that $t_1\, t_2 \to^* v_1\, t_2$. And, by Proposition B(2), because v_1 is a value and $t_2 \to^* v_2$, we have that $v_1\, t_2 \to^* v_1\, v_2$. Finally, because $t_1\, t_2 \to^* v_1\, t_2 \to^* v_1\, v_2 \to^* v$, we have that $t_1\, t_2 \to^* v$. Thus $t_1\, t_2$ converges and, since $t_1\, t_2 \to^* v$ and $v_1\, v_2 \to^* v$, it follows that

$$\overline{t_1\, t_2} = v = \overline{v_1\, v_2} = \overline{\overline{t_1}\, \overline{t_2}}.$$

Proposition F

Let P be the set of all $n \in \mathbb{N}$ such that, if $n \geq 2$, then, for all closed terms $t_1, \ldots, t_n$, if $t_1 \cdots t_n$ converges, then $t_1, \ldots, t_n$ and $\overline{t_1} \cdots \overline{t_n}$ converge and $\overline{t_1 \cdots t_n} = \overline{\overline{t_1} \cdots \overline{t_n}}$. It will suffice to use ordinary induction to prove that, for all $n \in \mathbb{N}$, $P(n)$.

- (Basis Step) $P(0)$ holds vacuously, because $0 \geq 2$ is false.
- (Inductive Step) Suppose $n \in \mathbb{N}$, and assume the inductive hypothesis, $P(n)$. We must show that $P(n+1)$. Suppose $n+1 \geq 2$, $t_1, \ldots, t_{n+1}$ are closed terms, and $t_1 \cdots t_{n+1}$ converges. We must show that $t_1, \ldots, t_{n+1}$ and $\overline{t_1} \cdots \overline{t_{n+1}}$ converge and $\overline{t_1 \cdots t_{n+1}} = \overline{\overline{t_1} \cdots \overline{t_{n+1}}}$. Since $n+1 \geq 2$, we have that $n \geq 1$. Hence there are two cases to consider.
  - Suppose $n = 1$. Thus $n+1 = 2$, so that t_1 and t_2 are closed terms, and $t_1\, t_2$ converges. We must show that t_1, t_2 and $\overline{t_1}\, \overline{t_2}$ converge, and that $\overline{t_1\, t_2} = \overline{\overline{t_1}\, \overline{t_2}}$, and this follows by Proposition D.
  - Suppose $n \geq 2$. Thus $t_1, \ldots, t_n, t_{n+1}$ are closed terms, and $t_1 \cdots t_n\, t_{n+1}$ converges. We must show that $t_1, \ldots, t_n, t_{n+1}$ and $\overline{t_1} \cdots \overline{t_n}\, \overline{t_{n+1}}$ converge and $\overline{t_1 \cdots t_n\, t_{n+1}} = \overline{\overline{t_1} \cdots \overline{t_n}\, \overline{t_{n+1}}}$. Because $(t_1 \cdots t_n)t_{n+1} = t_1 \cdots t_n\, t_{n+1}$ converges, Proposition D tells us that $t_1 \cdots t_n$, t_{n+1} and $\overline{t_1 \cdots t_n}\, \overline{t_{n+1}}$ converge and

    $$\overline{t_1 \cdots t_n\, t_{n+1}} = \overline{(t_1 \cdots t_n)t_{n+1}} = \overline{\overline{t_1 \cdots t_n}\, \overline{t_{n+1}}}.$$

    Because $n \geq 2$ and $t_1 \cdots t_n$ converges, the inductive hypothesis tells us that $t_1, \ldots, t_n$ and $\overline{t_1} \cdots \overline{t_n}$ converge and $\overline{t_1 \cdots t_n} = \overline{\overline{t_1} \cdots \overline{t_n}}$. So, we have that $t_1, \ldots, t_n$ and t_{n+1} converge. Because $\overline{t_1 \cdots t_n}\, \overline{t_{n+1}}$ converges, $\overline{t_1 \cdots t_n} = \overline{\overline{t_1} \cdots \overline{t_n}}$ and $\overline{t_{n+1}} = \overline{\overline{t_{n+1}}}$, it follows that $\overline{\overline{t_1} \cdots \overline{t_n}}\, \overline{\overline{t_{n+1}}}$ converges. Since $\overline{t_1} \cdots \overline{t_n}$, $\overline{t_{n+1}}$ and $\overline{\overline{t_1} \cdots \overline{t_n}}\, \overline{\overline{t_{n+1}}}$ converge, Proposition E tells us that $\overline{t_1} \cdots \overline{t_n}\, \overline{t_{n+1}} = (\overline{t_1} \cdots \overline{t_n})\overline{t_{n+1}}$ converges and

    $$\overline{\overline{t_1} \cdots \overline{t_n}\, \overline{t_{n+1}}} = \overline{(\overline{t_1} \cdots \overline{t_n})\overline{t_{n+1}}} = \overline{\overline{\overline{t_1} \cdots \overline{t_n}}\, \overline{\overline{t_{n+1}}}}.$$

    Since $\overline{t_1 \cdots t_n} = \overline{\overline{t_1} \cdots \overline{t_n}}$ and $\overline{t_{n+1}} = \overline{\overline{t_{n+1}}}$, it follows that

    $$\overline{\overline{t_1 \cdots t_n}\, \overline{t_{n+1}}} = \overline{\overline{\overline{t_1} \cdots \overline{t_n}}\, \overline{\overline{t_{n+1}}}} = \overline{\overline{t_1} \cdots \overline{t_n}\, \overline{t_{n+1}}}.$$

    Finally,

    $$\overline{t_1 \cdots t_n\, t_{n+1}} = \overline{\overline{t_1 \cdots t_n}\, \overline{t_{n+1}}} = \overline{\overline{t_1} \cdots \overline{t_n}\, \overline{t_{n+1}}}.$$

Proposition G

Let P be the set of all $n \in \mathbb{N}$ such that, if $n \geq 2$, then, for all closed terms $t_1, \ldots, t_n$, if $t_1, \ldots, t_n$ and $\overline{t_1} \cdots \overline{t_n}$ converge, then $t_1 \cdots t_n$ converges and $\overline{t_1 \cdots t_n} = \overline{\overline{t_1} \cdots \overline{t_n}}$. It will suffice to use ordinary induction to prove that, for all $n \in \mathbb{N}$, $P(n)$.

- (Basis Step) $P(0)$ holds vacuously, because $0 \geq 2$ is false.
- (Inductive Step) Suppose $n \in \mathbb{N}$ and assume the inductive hypothesis, $P(n)$. We must show that $P(n+1)$. Suppose $n+1 \geq 2$, $t_1, \ldots, t_{n+1}$ are closed terms, and $t_1, \ldots, t_{n+1}$ and $\overline{t_1} \cdots \overline{t_{n+1}}$ converge. We must show that $t_1 \cdots t_{n+1}$ converges and $\overline{t_1 \cdots t_{n+1}} = \overline{\overline{t_1} \cdots \overline{t_{n+1}}}$. Since $n+1 \geq 2$, we have that $n \geq 1$. Hence there are two cases to consider.
  - Suppose $n = 1$. Thus $n+1 = 2$, so that t_1 and t_2 are closed terms, t_1, t_2 and $\overline{t_1}\,\overline{t_2}$ converge. We must show that $t_1\,t_2$ converges and $\overline{t_1\,t_2} = \overline{\overline{t_1}\,\overline{t_2}}$, and this follows by Proposition E.
  - Suppose $n \geq 2$. Thus $t_1, \ldots, t_n, t_{n+1}$ are closed terms, and $t_1, \ldots, t_n, t_{n+1}$ and $\overline{t_1} \cdots \overline{t_n}\,\overline{t_{n+1}}$ converge. We must show that $t_1 \cdots t_n\,t_{n+1}$ converges and $\overline{t_1 \cdots t_n\,t_{n+1}} = \overline{\overline{t_1} \cdots \overline{t_n}\,\overline{t_{n+1}}}$. Because $(\overline{t_1} \cdots \overline{t_n})\overline{t_{n+1}} = \overline{t_1} \cdots \overline{t_n}\,\overline{t_{n+1}}$ converges, Proposition D tells us that $\overline{t_1} \cdots \overline{t_n}$, $\overline{t_{n+1}}$ and $\overline{\overline{t_1} \cdots \overline{t_n}}\,\overline{\overline{t_{n+1}}}$ converge, and

    $$\overline{\overline{t_1} \cdots \overline{t_n}\,\overline{t_{n+1}}} = \overline{(\overline{t_1} \cdots \overline{t_n})\overline{t_{n+1}}} = \overline{\overline{\overline{t_1} \cdots \overline{t_n}}\,\overline{\overline{t_{n+1}}}}.$$

    Since $n \geq 2$, and $t_1, \ldots, t_n$ and $\overline{t_1} \cdots \overline{t_n}$ converge, the inductive hypothesis tells us that $t_1 \cdots t_n$ converges and $\overline{t_1 \cdots t_n} = \overline{\overline{t_1} \cdots \overline{t_n}}$. So, we have that $t_1 \cdots t_n$ and t_{n+1} converge. Because $\overline{\overline{t_1} \cdots \overline{t_n}}\,\overline{\overline{t_{n+1}}}$ converges, $\overline{t_1 \cdots t_n} = \overline{\overline{t_1} \cdots \overline{t_n}}$ and $\overline{t_{n+1}} = \overline{\overline{t_{n+1}}}$, it follows that $\overline{t_1 \cdots t_n}\,\overline{t_{n+1}}$ converges. Since $t_1 \cdots t_n$, t_{n+1} and $\overline{t_1 \cdots t_n}\,\overline{t_{n+1}}$ converge, Proposition E tells us that $t_1 \cdots t_n\,t_{n+1} = (t_1 \cdots t_n)t_{n+1}$ converges and

    $$\overline{t_1 \cdots t_n\,t_{n+1}} = \overline{(t_1 \cdots t_n)t_{n+1}} = \overline{\overline{t_1 \cdots t_n}\,\overline{t_{n+1}}}.$$

    Since $\overline{t_1 \cdots t_n} = \overline{\overline{t_1} \cdots \overline{t_n}}$ and $\overline{t_{n+1}} = \overline{\overline{t_{n+1}}}$, we have that $\overline{\overline{t_1 \cdots t_n}\,\overline{t_{n+1}}} = \overline{\overline{\overline{t_1} \cdots \overline{t_n}}\,\overline{\overline{t_{n+1}}}}$. Hence, we have that

    $$\overline{t_1 \cdots t_n\,t_{n+1}} = \overline{\overline{t_1 \cdots t_n}\,\overline{t_{n+1}}} = \overline{\overline{\overline{t_1} \cdots \overline{t_n}}\,\overline{\overline{t_{n+1}}}} = \overline{\overline{t_1} \cdots \overline{t_n}\,\overline{t_{n+1}}}.$$