CIS 705 — Programming Languages — Spring 2009

Assignment 5

Model Answers

Exercise 1

For all closed terms s and s', if $s \to s'$, then one of the following conditions holds:

- **(E-App1)** there are closed terms t_1, t_1' and t_2 such that $s = t_1\,t_2$, $s' = t_1'\,t_2$ and $t_1 \to t_1'$;
- **(E-App2)** there are a closed value v_1 and closed terms t_2 and t_2' such that $s = v_1\,t_2$, $s' = v_1\,t_2'$ and $t_2 \to t_2'$; or
- **(E-AppAbs)** there are a variable x, a term t such that $\mathsf{FV}(t) \subseteq \{x\}$, and a closed value v such that $s = (\lambda x.\,t)v$ and $s' = [x \mapsto v]t$.

Exercise 2

Suppose P is a binary relation on closed terms. We sometimes write "$P(t, t')$" for "$(t, t') \in P$". The *principle of induction on the evaluation relation* says that,

$$\text{for all closed terms } t \text{ and } t', \text{ if } t \to t', \text{ then } P(t, t'),$$

follows from showing

- **(E-App1)** for all closed terms t_1, t_1' and t_2, if $t_1 \to t_1'$ and $P(t_1, t_1')$, then $P(t_1\,t_2, t_1'\,t_2)$;
- **(E-App2)** for all closed values v_1 and closed terms t_2 and t_2', if $t_2 \to t_2'$ and $P(t_2, t_2')$, then $P(v_1\,t_2, v_1\,t_2')$;
- **(E-AppAbs)** for all variables x, terms t such that $\mathsf{FV}(t) \subseteq \{x\}$, and closed values v, $P((\lambda x.\,t)v, [x \mapsto v]t)$.

Exercise 3

For all contexts Γ, terms s and types T, if $\Gamma \vdash s : T$, then one of the following conditions holds:

- **(T-Unit)** $s = \mathsf{unit}$ and $T = \mathsf{Unit}$;
- **(T-Var)** there is a variable x such that $s = x$ and $(x, T) \in \Gamma$;
- **(T-Abs)** there are a variable x, a term t and types T_1 and T_2 such that $s = \lambda x.\,t$, $T = T_1 \to T_2$ and $\Gamma[x \mapsto T_1] \vdash t : T_2$; or
- **(T-App)** there are terms t_1 and t_2 and types T_1 and T_2 such that $s = t_1\,t_2$, $T = T_2$, $\Gamma \vdash t_1 : T_1 \to T_2$ and $\Gamma \vdash t_2 : T_1$.

Exercise 4

Suppose P is a ternary relation between contexts, terms and types. We sometimes write "$P(\Gamma, t, T)$" for "$(\Gamma, t, T) \in P$". The *principle of induction on the typing relation* says that,

$$\text{for all contexts } \Gamma, \text{ terms } t \text{ and types } T, \text{ if } \Gamma \vdash t : T, \text{ then } P(\Gamma, t, T)$$

follows from showing

(T-Unit) for all contexts Γ, $P(\Gamma, \mathsf{unit}, \mathsf{Unit})$;

(T-Var) for all contexts Γ, variables x and types T, if $(x, T) \in \Gamma$, then $P(\Gamma, x, T)$;

(T-Abs) for all contexts Γ, variables x, terms t, and types T_1 and T_2, if $\Gamma[x \mapsto T_1] \vdash t : T_2$ and $P(\Gamma[x \mapsto T_1], t, T_2)$, then $P(\Gamma, \lambda x.\, t, T_1 \to T_2)$;

(T-App) for all contexts Γ, terms t_1 and t_2, and types T_1 and T_2, if $\Gamma \vdash t_1 : T_1 \to T_2$, $\Gamma \vdash t_2 : T_1$, $P(\Gamma, t_1, T_1 \to T_2)$ and $P(\Gamma, t_2, T_1)$, then $P(\Gamma, t_1\, t_2, T_2)$.

Exercise 5

Let $P = \{ (\Gamma, t, T) \mid \Gamma \text{ is a context}, t \text{ is a term}, T \text{ is a type and } \mathsf{FV}(t) \subseteq \mathsf{dom}(\Gamma) \}$. We use induction on the typing relation to prove that, for all contexts Γ, terms t and types T, if $\Gamma \vdash t : T$, then $P(\Gamma, t, T)$.

(T-Unit) Suppose Γ is a context. We must show that $P(\Gamma, \mathsf{unit}, \mathsf{Unit})$, and this follows since $\mathsf{FV}(\mathsf{unit}) = \emptyset \subseteq \mathsf{dom}(\Gamma)$.

(T-Var) Suppose Γ is a context, x is a variable, T is a type and $(x, T) \in \Gamma$. We must show that $P(\Gamma, x, T)$, and this follows since $\mathsf{FV}(x) = \{x\} \subseteq \mathsf{dom}(\Gamma)$.

(T-Abs) Suppose Γ is a context, x is a variable, t is a term, T_1 and T_2 are types, and $\Gamma[x \mapsto T_1] \vdash t : T_2$, and assume the inductive hypothesis, $P(\Gamma[x \mapsto T_1], t, T_2)$. We must show that $P(\Gamma, \lambda x.\, t, T_1 \to T_2)$. By the inductive hypothesis, we have that $\mathsf{FV}(t) \subseteq \mathsf{dom}(\Gamma[x \mapsto T_1]) = \mathsf{dom}(\Gamma) \cup \{x\}$. Suppose $y \in \mathsf{FV}(\lambda x.\, t)$. Since $\mathsf{FV}(\lambda x.\, t) = \mathsf{FV}(t) \setminus \{x\}$, it follows that $y \in \mathsf{FV}(t)$ and $y \neq x$. Thus $y \in \mathsf{FV}(t) \subseteq \mathsf{dom}(\Gamma) \cup \{x\}$. But $y \neq x$, and thus $y \in \mathsf{dom}(\Gamma)$, as required.

(T-App) Suppose Γ is a context, t_1 and t_2 are terms, T_1 and T_2 are types, $\Gamma \vdash t_1 : T_1 \to T_2$ and $\Gamma \vdash t_2 : T_1$, and assume the inductive hypothesis, $P(\Gamma, t_1, T_1 \to T_2)$ and $P(\Gamma, t_2, T_1)$. We must show that $P(\Gamma, t_1\, t_2, T_2)$. By the inductive hypothesis, we have that $\mathsf{FV}(t_1) \subseteq \mathsf{dom}(\Gamma)$ and $\mathsf{FV}(t_2) \subseteq \mathsf{dom}(\Gamma)$. Thus $\mathsf{FV}(t_1\, t_2) = \mathsf{FV}(t_1) \cup \mathsf{FV}(t_2) \subseteq \mathsf{dom}(\Gamma) \cup \mathsf{dom}(\Gamma) = \mathsf{dom}(\Gamma)$.

Exercise 6

Suppose v is a closed value.

- Suppose $\emptyset \vdash v : \mathsf{Unit}$. By inversion on the typing relation, there are four cases to consider.

  (T-Unit) Suppose $v = \mathsf{unit}$ and $\mathsf{Unit} = \mathsf{Unit}$. Then $v = \mathsf{unit}$.

(T-Var) Suppose there is a variable x such that $v = x$ and $(x, \mathsf{Unit}) \in \Gamma$. But this contradicts the fact that v is closed. Thus $v = \mathsf{unit}$.

(T-Abs) Suppose there are a variable x, a term t and types T_1 and T_2 such that $v = \lambda x. t$, $\mathsf{Unit} = T_1 \to T_2$ and $\Gamma[x \mapsto T_1] \vdash t : T_2$. But $\mathsf{Unit} = T_1 \to T_2$ is a contradiction, and thus $v = \mathsf{unit}$.

(T-App) Suppose there are terms t_1 and t_2 and types T_1 and T_2 such that $v = t_1\, t_2$, $\mathsf{Unit} = T_2$, $\Gamma \vdash t_1 : T_1 \to T_2$ and $\Gamma \vdash t_2 : T_1$. Thus $t_1\, t_2 = v$ is a value—contradiction. Hence $v = \mathsf{unit}$.

- Suppose $\emptyset \vdash v : S_1 \to S_2$ for some types S_1 and S_2. By inversion on the typing relation, there are four cases to consider.

(T-Unit) Suppose $v = \mathsf{unit}$ and $S_1 \to S_2 = \mathsf{Unit}$. But $S_1 \to S_2 = \mathsf{Unit}$ is a contradiction, and thus v is an abstraction.

(T-Var) Suppose there is a variable x such that $v = x$ and $(x, S_1 \to S_2) \in \Gamma$. But this contradicts the fact that v is closed. Thus v is an abstraction.

(T-Abs) Suppose there are a variable x, a term t and types T_1 and T_2 such that $v = \lambda x. t$, $S_1 \to S_2 = T_1 \to T_2$ and $\Gamma[x \mapsto T_1] \vdash t : T_2$. Thus v is an abstraction.

(T-App) Suppose there are terms t_1 and t_2 and types T_1 and T_2 such that $v = t_1\, t_2$, $S_1 \to S_2 = T_2$, $\Gamma \vdash t_1 : T_1 \to T_2$ and $\Gamma \vdash t_2 : T_1$. Thus $t_1\, t_2 = v$ is a value—contradiction. Hence v is an abstraction.

Exercise 7

We prove the proposition. Let $P = \{\, (t, t') \mid t$ and t' are closed terms, and, for all closed terms t'', if $t \to t''$, then $t' = t''\,\}$. We use induction on the evaluation relation to show that, for all closed terms t and t', if $t \to t'$, then $P(t, t')$. (Then, suppose t, t' and t'' are closed terms, $t \to t'$ and $t \to t''$. Thus $P(t, t')$, so that, if $t \to t''$, then $t' = t''$. But $t \to t''$, and thus $t' = t''$.)

(E-App1) Suppose t_1, t_1' and t_2 are closed terms, and $t_1 \to t_1'$, and assume the inductive hypothesis, $P(t_1, t_1')$. We must show that $P(t_1\, t_2, t_1'\, t_2)$. Suppose u is a closed term and $t_1\, t_2 \to u$. Because $t_1 \to t_1'$, we have that t_1 is not a normal form, and thus that t_1 is not a value. Thus, by inversion of the evaluation relation, only (E-App1) applies, and we have that there is a closed term t_1'' such that $u = t_1''\, t_2$ and $t_1 \to t_1''$. By the inductive hypothesis, it follows that $t_1' = t_1''$. Thus $t_1'\, t_2 = t_1''\, t_2 = u$.

(E-App2) Suppose v_1 is a closed value, t_2 and t_2' are closed terms, and $t_2 \to t_2'$, and assume the inductive hypothesis, $P(t_2, t_2')$. We must show that $P(v_1\, t_2, v_1\, t_2')$. Suppose u is a closed term and $v_1\, t_2 \to u$. Because $t_2 \to t_2'$, we have that t_2 is not a normal form, and thus that t_2 is not a value. Furthermore, v_1 is a value, and thus is a normal form. Hence, by inversion of the evaluation relation, only (E-App2) applies, and we have that there is a closed term t_2'' such that $u = v_1\, t_2''$ and $t_2 \to t_2''$. By the inductive hypothesis, it follows that $t_2' = t_2''$. Thus $v_1\, t_2' = v_1 t_2'' = u$.

(E-AppAbs) Suppose x is a variable, t is a term such that $\mathsf{FV}(t) \subseteq \{x\}$, and v is a value. We must show that $P((\lambda x.\, t)v, [x \mapsto v]t)$. Suppose u is a closed term such that $(\lambda x.\, t)v \to u$. Because $\lambda x.\, t$ is a value, it is a normal form. And, because v is a value, it is a normal form. Thus, by inversion of the evaluation relation, only (E-AppAbs) applies, and we have that $u = [x \mapsto v]t$, so that $[x \mapsto v]t = u$.

Exercise 8

We disprove the proposition. Let x be a variable. We will show that $\emptyset \vdash \lambda x.\, x : \mathsf{Unit} \to \mathsf{Unit}$ and $\emptyset \vdash \lambda x.\, x : (\mathsf{Unit} \to \mathsf{Unit}) \to (\mathsf{Unit} \to \mathsf{Unit})$.

- Because $(x, \mathsf{Unit}) \in \{(x, \mathsf{Unit})\}$, (T-Var) tells us that $\{(x, \mathsf{Unit})\} \vdash x : \mathsf{Unit}$. But $\{(x, \mathsf{Unit})\} = \emptyset[x \mapsto \mathsf{Unit}]$, and thus $\emptyset[x \mapsto \mathsf{Unit}] \vdash x : \mathsf{Unit}$. Thus $\emptyset \vdash \lambda x.\, x : \mathsf{Unit} \to \mathsf{Unit}$, by (T-Abs).
- Because $(x, \mathsf{Unit} \to \mathsf{Unit}) \in \{(x, \mathsf{Unit} \to \mathsf{Unit})\}$, (T-Var) tells us that $\{(x, \mathsf{Unit} \to \mathsf{Unit})\} \vdash x : \mathsf{Unit} \to \mathsf{Unit}$. But $\{(x, \mathsf{Unit} \to \mathsf{Unit})\} = \emptyset[x \mapsto \mathsf{Unit} \to \mathsf{Unit}]$, and thus $\emptyset[x \mapsto \mathsf{Unit} \to \mathsf{Unit}] \vdash x : \mathsf{Unit} \to \mathsf{Unit}$. Thus $\emptyset \vdash \lambda x.\, x : (\mathsf{Unit} \to \mathsf{Unit}) \to (\mathsf{Unit} \to \mathsf{Unit})$, by (T-Abs).

Exercise 9

We prove the theorem. Let $P = \{\, (\Gamma, t, T) \mid \Gamma \text{ is a context}, t \text{ is a term}, T \text{ is a type and, if } \Gamma = \emptyset, \text{ then } t \text{ is a closed term that is not stuck} \,\}$. We use induction on the typing relation to prove that, for all contexts Γ, terms t and types T, if $\Gamma \vdash t : T$, then $P(\Gamma, t, T)$. (Then, suppose t is a closed term, and t is well-typed. Thus $\emptyset \vdash t : T$ for some type T. By the result of the induction, we have that $P(\emptyset, t, T)$. Thus, if $\emptyset = \emptyset$, then t is a closed term that is not stuck. But $\emptyset = \emptyset$, and thus t is not stuck.)

(T-Unit) Suppose Γ is a context. We must show that $P(\Gamma, \mathsf{unit}, \mathsf{Unit})$, and this follows because unit is a closed value, and so is a closed term that is not stuck.

(T-Var) Suppose Γ is a context, x is a variable, T is a type and $(x, T) \in \Gamma$. We must show that $P(\Gamma, x, T)$. Suppose $\Gamma = \emptyset$. We must show that x is a closed term that is not stuck. But $(x, T) \in \Gamma = \emptyset$—contradiction. Thus x is a closed term that is not stuck.

(T-Abs) Suppose Γ is a context, x is a variable, t is a term, T_1 and T_2 are types, and $\Gamma[x \mapsto T_1] \vdash t : T_2$, and assume the inductive hypothesis, $P(\Gamma[x \mapsto T_1], t, T_2)$. We must show that $P(\Gamma, \lambda x.\, t, T_1 \to T_2)$. Suppose $\Gamma = \emptyset$. We must show that $\lambda x.\, t$ is a closed term that is not stuck. Because $\emptyset[x \mapsto T_1] \vdash t : T_2$, the Free Variables Lemma (Exercise 5) tells us that $\mathsf{FV}(t) \subseteq \mathsf{dom}(\emptyset[x \mapsto T_1]) = \mathsf{dom}(\emptyset) \cup \{x\} = \emptyset \cup \{x\} = \{x\}$. Thus $\mathsf{FV}(\lambda x.\, t) = \mathsf{FV}(t) \setminus \{x\} \subseteq \{x\} \setminus \{x\} = \emptyset$, showing that $\lambda x.\, t$ is closed. And $\lambda x.\, t$ is a value, showing that $\lambda x.\, t$ is not stuck. Thus $\lambda x.\, t$ is a closed term that is not stuck.

(T-App) Suppose Γ is a context, t_1 and t_2 are terms, T_1 and T_2 are types, $\Gamma \vdash t_1 : T_1 \to T_2$ and $\Gamma \vdash t_2 : T_1$, and assume the inductive hypothesis, $P(\Gamma, t_1, T_1 \to T_2)$ and $P(\Gamma, t_2, T_1)$. We must show that $P(\Gamma, t_1\, t_2, T_2)$. Suppose $\Gamma = \emptyset$. We must show that $t_1\, t_2$ is a closed term that is not stuck. By $P(\Gamma, t_1, T_1 \to T_2)$, we have that t_1 is a closed term that is not stuck. And, by $P(\Gamma, t_2, T_1)$, t_2 is a closed term that is not stuck. Thus $t_1\, t_2$ is closed. Because t_1 is a closed term that is not stuck, there are two cases to consider.

- Suppose t_1 is not a normal form. Thus there is a closed term t'_1 such that $t_1 \rightarrow t'_1$. Hence $t_1\, t_2 \rightarrow t'_1\, t_2$, by (E-App1), showing that $t_1\, t_2$ is not a normal form, and so is not stuck. Thus $t_1\, t_2$ is a closed term that is not stuck.
- Suppose t_1 is a closed value. Because t_2 is a closed term that is not stuck, there are two subcases to consider.
  - Suppose t_2 is not a normal form. Thus there is a closed term t'_2 such that $t_2 \rightarrow t'_2$. Hence $t_1\, t_2 \rightarrow t_1\, t'_2$, by (E-App2), because t_1 is a value, showing that $t_1\, t_2$ is not a normal form, and so is not stuck. Thus $t_1\, t_2$ is a closed term that is not stuck.
  - Suppose t_2 is closed value. Because t_1 is a closed value and $\emptyset \vdash t_1 : T_1 \rightarrow T_2$, the Canonical Forms Lemma (Exercise 6) tells us that t_1 is an abstraction. Thus $t_1 = \lambda x.\, u$ for a variable x and a term u. Because t_2 is a value, we have that $t_1\, t_2 = (\lambda x.\, u)t_2 \rightarrow [x \mapsto t_2]u$, by (E-AppAbs), showing that $t_1\, t_2$ is not a normal form, and so is not stuck. Thus $t_1\, t_2$ is a closed term that is not stuck.

Exercise 10

We prove the theorem. First, we prove two supporting lemmas.

Lemma 10.1 (Weakening)
For all contexts Γ *and* Γ'*, terms* t *and types* T*, if* $\Gamma \vdash t : T$ *and* $\Gamma \subseteq \Gamma'$*, then* $\Gamma' \vdash t : T$.

Proof. Let $P = \{\, (\Gamma, t, T) \mid \Gamma \text{ is a context}, t \text{ is a term}, T \text{ is a type and, for all contexts } \Gamma', \text{ if } \Gamma \subseteq \Gamma', \text{ then } \Gamma' \vdash t : T \,\}$. We use induction on the typing relation to prove that, for all contexts Γ, terms t and types T, if $\Gamma \vdash t : T$, then $P(\Gamma, t, T)$. (Then, suppose Γ and Γ' are contexts, t is a term, T is a type, $\Gamma \vdash t : T$ and $\Gamma \subseteq \Gamma'$. By the result of the induction, $P(\Gamma, t, T)$. Thus, since $\Gamma \subseteq \Gamma'$, we have that $\Gamma' \vdash t : T$.)

(T-Unit) Suppose Γ is a context. We must show that $P(\Gamma, \mathsf{unit}, \mathsf{Unit})$. Suppose Γ' is a context and $\Gamma \subseteq \Gamma'$. We must show that $\Gamma' \vdash \mathsf{unit} : \mathsf{Unit}$, and this follows by (T-Unit).

(T-Var) Suppose Γ is a context, x is a variable, T is a type and $(x, T) \in \Gamma$. We must show that $P(\Gamma, x, T)$. Suppose Γ' is a context and $\Gamma \subseteq \Gamma'$. We must show that $\Gamma' \vdash x : T$, and this follows by (T-Var), since $(x, T) \in \Gamma \subseteq \Gamma'$.

(T-Abs) Suppose Γ is a context, x is a variable, t is a term, T_1 and T_2 are types, and $\Gamma[x \mapsto T_1] \vdash t : T_2$, and assume the inductive hypothesis, $P(\Gamma[x \mapsto T_1], t, T_2)$. We must show that $P(\Gamma, \lambda x.\, t, T_1 \rightarrow T_2)$. Suppose Γ' is a context and $\Gamma \subseteq \Gamma'$. We must show that $\Gamma' \vdash \lambda x.\, t : T_1 \rightarrow T_2$. Because of (T-Abs), it will suffice to show that $\Gamma'[x \mapsto T_1] \vdash t : T_2$. Thus, by the inductive hypothesis, it will suffice to show that $\Gamma[x \mapsto T_1] \subseteq \Gamma'[x \mapsto T_1]$. Suppose $(y, T) \in \Gamma[x \mapsto T_1]$. We must show that $(y, T) \in \Gamma'[x \mapsto T_1]$. There are two cases to consider.

- Suppose $y = x$. Because $(y, T) \in \Gamma[x \mapsto T_1]$, we have that $\Gamma[x \mapsto T_1](y) = T$. But $y = x$, and thus $\Gamma[x \mapsto T_1](y) = T_1$, so that $T = T_1$. Thus $(y, T) = (x, T_1) \in \Gamma'[x \mapsto T_1]$.
- Suppose $y \neq x$. Since $\Gamma[x \mapsto T_1](y) = T$ and $y \neq x$, it follows that $T = \Gamma[x \mapsto T_1](y) = \Gamma(y)$, so that $(y, T) \in \Gamma \subseteq \Gamma'$. Thus $\Gamma'[x \mapsto T_1](y) = \Gamma'(y) = T$, showing that $(y, T) \in \Gamma'[x \mapsto T_1]$.

(T-App) Suppose Γ is a context, t_1 and t_2 are terms, T_1 and T_2 are types, $\Gamma \vdash t_1 : T_1 \to T_2$ and $\Gamma \vdash t_2 : T_1$, and assume the inductive hypothesis, $P(\Gamma, t_1, T_1 \to T_2)$ and $P(\Gamma, t_2, T_1)$. We must show that $P(\Gamma, t_1\, t_2, T_2)$. Suppose Γ' is a context and $\Gamma \subseteq \Gamma'$. We must show that $\Gamma' \vdash t_1\, t_2 : T_2$. By the inductive hypothesis, we have that $\Gamma' \vdash t_1 : T_1 \to T_2$ and $\Gamma' \vdash t_2 : T_1$. Thus, by (T-App), it follows that $\Gamma' \vdash t_1\, t_2 : T_2$.

□

Lemma 10.2 (Typing of Substitutions)
For all contexts Γ, variables y, terms t, values v and types T, T', if $\Gamma[y \mapsto T'] \vdash t : T$ and $\emptyset \vdash v : T'$, then $\Gamma \vdash [y \mapsto v]t : T$.

Proof. It suffices to show that, for all variables y, values v and types T', if $\emptyset \vdash v : T'$, then, for all contexts Γ, terms t and types T, if $\Gamma[y \mapsto T'] \vdash t : T$, then $\Gamma \vdash [y \mapsto v]t : T$.

Suppose y is a variable, v is a value, T' is a type, and $\emptyset \vdash v : T'$. We must show that, for all contexts Γ, terms t and types T, if $\Gamma[y \mapsto T'] \vdash t : T$, then $\Gamma \vdash [y \mapsto v]t : T$.

Let $P = \{\, (\Gamma, t, T) \mid \Gamma \text{ is a context}, t \text{ is a term}, T \text{ is a type and, for all contexts } \Gamma', \text{ if } \Gamma = \Gamma'[y \mapsto T'], \text{ then } \Gamma' \vdash [y \mapsto v]t : T \,\}$. We use induction on the typing relation to prove that, for all contexts Γ, terms t and types T, if $\Gamma \vdash t : T$, then $P(\Gamma, t, T)$. (Then, suppose Γ is a context, t is a term, T is a type and $\Gamma[y \mapsto T'] \vdash t : T$. We must show that $\Gamma \vdash [y \mapsto v]t : T$. By the result of the induction, we have that $P(\Gamma[y \mapsto T'], t, T)$, so that for all contexts Γ', if $\Gamma[y \mapsto T'] = \Gamma'[y \mapsto T']$, then $\Gamma' \vdash [y \mapsto v]t : T$. Thus $\Gamma \vdash [y \mapsto v]t : T$.)

(T-Unit) Suppose Γ is a context. We must show that $P(\Gamma, \mathsf{unit}, \mathsf{Unit})$. Suppose Γ' is a context and $\Gamma = \Gamma'[y \mapsto T']$. We must show that $\Gamma' \vdash [y \mapsto v]\mathsf{unit} : \mathsf{Unit}$. Since $[y \mapsto v]\mathsf{unit} = \mathsf{unit}$, it will suffice to show $\Gamma' \vdash \mathsf{unit} : \mathsf{Unit}$, which follows by (T-Unit).

(T-Var) Suppose Γ is a context, x is a variable, T is a type and $(x, T) \in \Gamma$. We must show that $P(\Gamma, x, T)$. Suppose Γ' is a context and $\Gamma = \Gamma'[y \mapsto T']$. We must show that $\Gamma' \vdash [y \mapsto v]x : T$. There are two cases to consider.

- Suppose $y = x$. Thus $[y \mapsto v]x = v$, so we must show that $\Gamma' \vdash v : T$. Because $\emptyset \vdash v : T'$, the Weakening Lemma (Lemma 10.1), tells us that $\Gamma' \vdash v : T'$. Because $(x, T) \in \Gamma = \Gamma'[y \mapsto T']$ and $x = y$, we have that $T = \Gamma'[y \mapsto T'](x) = T'$. Thus $\Gamma' \vdash v : T$, as required.
- Suppose $y \neq x$. Thus $[y \mapsto v]x = x$, so we must show that $\Gamma' \vdash x : T$. Because $(x, T) \in \Gamma = \Gamma'[y \mapsto T']$ and $x \neq y$, we have that $T = \Gamma'[y \mapsto T'](x) = \Gamma'(x)$. Thus $(x, T) \in \Gamma'$, so that $\Gamma' \vdash x : T$ follows by (T-Var).

(T-Abs) Suppose Γ is a context, x is a variable, t is a term, T_1 and T_2 are types, and $\Gamma[x \mapsto T_1] \vdash t : T_2$, and assume the inductive hypothesis, $P(\Gamma[x \mapsto T_1], t, T_2)$. We must show that $P(\Gamma, \lambda x.\, t, T_1 \to T_2)$. Suppose Γ' is a context and $\Gamma = \Gamma'[y \mapsto T']$. We must show that $\Gamma' \vdash [y \mapsto v]\lambda x.\, t : T_1 \to T_2$. There are two cases to consider.

- Suppose $x = y$. Then $[y \mapsto v]\lambda x.\, t = \lambda x.\, t$, so it will suffice to show that $\Gamma' \vdash \lambda x.\, t : T_1 \to T_2$. By (T-Abs), it will suffice to show that $\Gamma'[x \mapsto T_1] \vdash t : T_2$. Because $\Gamma[x \mapsto T_1] \vdash t : T_2$ and $\Gamma = \Gamma'[y \mapsto T']$, we have that $\Gamma'[y \mapsto T'][x \mapsto T_1] \vdash t : T_2$. Thus it will suffice to show that $\Gamma'[x \mapsto T_1] = \Gamma'[y \mapsto T'][x \mapsto T_1]$. Because $x = y$, both of these contexts have domain $\mathsf{dom}(\Gamma') \cup \{x\}$, both yield T_1 when called with x, and both yield $\Gamma'(z)$ when called with some $z \in \mathsf{dom}(\Gamma') \setminus \{x\}$. Thus, they are equal.

- Suppose $x \neq y$. Then $[y \mapsto v]\lambda x.\, t = \lambda x.\, [y \mapsto v]t$, so it will suffice to show that $\Gamma' \vdash \lambda x.\, [y \mapsto v]t : T_1 \to T_2$. By (T-Abs), it will suffice to show that $\Gamma'[x \mapsto T_1] \vdash [y \mapsto v]t : T_2$. By the inductive hypothesis, we have that, for all contexts Γ', if $\Gamma[x \mapsto T_1] = \Gamma'[y \mapsto T']$, then $\Gamma' \vdash [y \mapsto v]t : T_2$. Thus, if $\Gamma[x \mapsto T_1] = \Gamma'[x \mapsto T_1][y \mapsto T']$, then $\Gamma'[x \mapsto T_1] \vdash [y \mapsto v]t : T_2$. Thus, it will suffice to show that $\Gamma[x \mapsto T_1] = \Gamma'[x \mapsto T_1][y \mapsto T']$. Because $\Gamma = \Gamma'[y \mapsto T']$, it will suffice to show that $\Gamma'[y \mapsto T'][x \mapsto T_1] = \Gamma'[x \mapsto T_1][y \mapsto T']$. The contexts both have domain $\mathsf{dom}(\Gamma') \cup \{x\} \cup \{y\}$, both yield T_1 when called with x, both yield T' when called with y, and both yield $\Gamma'(z)$ when called with a $z \in \mathsf{dom}(\Gamma') \setminus \{x, y\}$. Thus, they are equal.

(T-App) Suppose Γ is a context, t_1 and t_2 are terms, T_1 and T_2 are types, $\Gamma \vdash t_1 : T_1 \to T_2$ and $\Gamma \vdash t_2 : T_1$, and assume the inductive hypothesis, $P(\Gamma, t_1, T_1 \to T_2)$ and $P(\Gamma, t_2, T_1)$. We must show that $P(\Gamma, t_1\, t_2, T_2)$. Suppose Γ' is a context and $\Gamma = \Gamma'[y \mapsto T']$. We must show that $\Gamma' \vdash [y \mapsto v](t_1\, t_2) : T_2$. Since $[y \mapsto v](t_1\, t_2) = [y \mapsto v]t_1\, [y \mapsto v]t_2$, we must show that $\Gamma' \vdash [y \mapsto v]t_1\, [y \mapsto v]t_2 : T_2$. By $P(\Gamma, t_1, T_1 \to T_2)$, we have that $\Gamma' \vdash [y \mapsto v]t_1 : T_1 \to T_2$. And, by $P(\Gamma, t_2, T_1)$, we have that $\Gamma' \vdash [y \mapsto v]t_2 : T_1$. Thus, by T-App, we have that $\Gamma' \vdash [y \mapsto v]t_1\, [y \mapsto v]t_2 : T_2$.

□

Now, we prove the Preservation Theorem. Let $P = \{\, (t, t') \mid t$ and t' are closed terms, and, for all types T, if $\emptyset \vdash t : T$, then $\emptyset \vdash t' : T \,\}$. We use induction on the evaluation relation to show that, for all closed terms t and t', if $t \to t'$, then $P(t, t')$. (Then, suppose t and t' are closed terms, T is a type, $\emptyset \vdash t : T$ and $t \to t'$. By the result of the induction, we have that $P(t, t')$. But $\emptyset \vdash t : T$, and thus $\emptyset \vdash t' : T$.)

(E-App1) Suppose t_1, t_1' and t_2 are closed terms, and $t_1 \to t_1'$, and assume the inductive hypothesis, $P(t_1, t_1')$. We must show that $P(t_1\, t_2, t_1'\, t_2)$. Suppose T is a type and $\emptyset \vdash t_1\, t_2 : T$. We must show that $\emptyset \vdash t_1'\, t_2 : T$. By inversion of the typing relation, we have that there is a type T' such that $\emptyset \vdash t_1 : T' \to T$ and $\emptyset \vdash t_2 : T'$. By the inductive hypothesis, it follows that $\emptyset \vdash t_1' : T' \to T$. Thus by (T-App), we have that $\emptyset \vdash t_1'\, t_2 : T$.

(E-App2) Suppose v_1 is a closed value, t_2 and t_2' are closed terms, and $t_2 \to t_2'$, and assume the inductive hypothesis, $P(t_2, t_2')$. We must show that $P(v_1\, t_2, v_1\, t_2')$. Suppose T is a type and $\emptyset \vdash v_1\, t_2 : T$. We must show that $\emptyset \vdash v_1\, t_2' : T$. By inversion of the typing relation, we have that there is a type T' such that $\emptyset \vdash v_1 : T' \to T$ and $\emptyset \vdash t_2 : T'$. By the inductive hypothesis, it follows that $\emptyset \vdash t_2' : T'$. Thus by (T-App), we have that $\emptyset \vdash v_1\, t_2' : T$.

(E-AppAbs) Suppose x is a variable, t is a term such that $\mathsf{FV}(t) \subseteq \{x\}$, and v is a value. We must show that $P((\lambda x.\, t)v, [x \mapsto v]t)$. Suppose T is a type and $\emptyset \vdash (\lambda x.\, t)v : T$. We must show that $\emptyset \vdash [x \mapsto v]t : T$. By the inversion of the typing relation, we have that there is a type T' such that $\emptyset \vdash \lambda x.\, t : T' \to T$ and $\emptyset \vdash v : T'$. Because $\emptyset \vdash \lambda x.\, t : T' \to T$, by another inversion of the typing relation, we have that $\emptyset[x \mapsto T'] \vdash t : T$. Since $\emptyset[x \mapsto T'] \vdash t : T$ and $\emptyset \vdash v : T'$, Lemma 10.2 tells us that $\emptyset \vdash [x \mapsto v]t : T$.