CIS 705 — Programming Languages — Spring 2009

Assignment 1 (100 Points)

Due by 2:30 p.m. on Thursday, February 12

The context for this assignment is Chapters 2-3 of *TAPL*.

Definition of Nats

Let Nats be the smallest set such that:

(Zero)

$$\mathsf{zero} \in \mathsf{Nats};$$

(Succ) for all t,

$$\frac{t \in \mathsf{Nats}}{\mathsf{succ}\ t \in \mathsf{Nats}.}$$

(Here, zero and succ are constructors, so that, for all t, $\mathsf{zero} \neq \mathsf{succ}\ t$, and, for all t_1, t_2, $\mathsf{succ}\ t_1 = \mathsf{succ}\ t_2$ iff $t_1 = t_2$.)

The *inversion lemma* for Nats says that, for all $s \in \mathsf{Nats}$, either:

(Zero) $s = \mathsf{zero}$; or

(Succ) $s = \mathsf{succ}\ t$, for some $t \in \mathsf{Nats}$.

Suppose $P \subseteq \mathsf{Nats}$. The *principle of structural induction on* Nats says that,

$$\text{for all } t \in \mathsf{Nats}, P(t)$$

follows from showing

(Zero) $P(\mathsf{zero})$;

(Succ) for all $t \in \mathsf{Nats}$, if (†) $P(t)$, then $P(\mathsf{succ}\ t)$.

We refer to (†) as the *inductive hypothesis*.

Definition of Plus

Let Plus be the smallest subset of $\mathsf{Nats} \times \mathsf{Nats} \times \mathsf{Nats}$ such that:

(Plus-Zero) for all $t \in \mathsf{Nats}$,

$$(\mathsf{zero}, t, t) \in \mathsf{Plus};$$

(Plus-Succ) for all $t_1, t_2, t_3 \in \mathsf{Nats}$,

$$\frac{(t_1, t_2, t_3) \in \mathsf{Plus}}{(\mathsf{succ}\ t_1, t_2, \mathsf{succ}\ t_3) \in \mathsf{Plus}.}$$

The *inversion lemma* for Plus says that, for all $s_1, s_2, s_3 \in \mathsf{Nats}$, if $(s_1, s_2, s_3) \in \mathsf{Plus}$, then either:

(Plus-Zero) there is a $t \in \mathsf{Nats}$ such that $s_1 = \mathsf{zero}$, $s_2 = t$ and $s_3 = t$ (so that $s_2 = s_3$); or

(Plus-Succ) there are $t_1, t_2, t_3 \in \mathsf{Nats}$ such that $s_1 = \mathsf{succ}\ t_1$, $s_2 = t_2$, $s_3 = \mathsf{succ}\ t_3$ and $(t_1, t_2, t_3) \in \mathsf{Plus}$ (so that $(t_1, s_2, t_3) \in \mathsf{Plus}$).

Suppose $P \subseteq \mathsf{Nats} \times \mathsf{Nats} \times \mathsf{Nats}$. We sometimes write "$P(t_1, t_2, t_3)$" for "$(t_1, t_2, t_3) \in P$". The *principle of induction on* Plus says that,

$$\text{for all } t_1, t_2, t_3 \in \mathsf{Nats}, \text{ if } (t_1, t_2, t_3) \in \mathsf{Plus}, \text{ then } P(t_1, t_2, t_3),$$

follows from showing

(Plus-Zero) for all $t \in \mathsf{Nats}$, $P(\mathsf{zero}, t, t)$;

(Plus-Succ) for all $t_1, t_2, t_3 \in \mathsf{Nats}$, if $(t_1, t_2, t_3) \in \mathsf{Plus}$ and (†) $P(t_1, t_2, t_3)$, then $P(\mathsf{succ}\ t_1, t_2, \mathsf{succ}\ t_3)$.

We refer to (†) as the *inductive hypothesis*.

Exercise 1 (25 Points)

Prove that, for all $t_1, t_2 \in \mathsf{Nats}$, there is a $t_3 \in \mathsf{Nats}$ such that $(t_1, t_2, t_3) \in \mathsf{Plus}$.

Exercise 2 (25 Points)

Prove that, for all $t_1, t_2, t_3, t_3' \in \mathsf{Nats}$, if $(t_1, t_2, t_3) \in \mathsf{Plus}$ and $(t_1, t_2, t_3') \in \mathsf{Plus}$, then $t_3 = t_3'$. (This completes the proof that, for all $t_1, t_2 \in \mathsf{Nats}$, there is a unique $t_3 \in \mathsf{Nats}$ such that $(t_1, t_2, t_3) \in \mathsf{Plus}$.)

Exercise 3 (25 Points)

Prove that, for all $t_1, t_2, t_3, s_1, s_2, s_3 \in \mathsf{Nats}$, if $(t_1, t_2, s_1) \in \mathsf{Plus}$, $(s_1, t_3, s_2) \in \mathsf{Plus}$ and $(t_2, t_3, s_3) \in \mathsf{Plus}$, then $(t_1, s_3, s_2) \in \mathsf{Plus}$.

Exercise 4 (25 Points)

Prove that, for all $t_1, t_2, t_3 \in \mathsf{Nats}$, if $(t_1, t_2, t_3) \in \mathsf{Plus}$, then $(t_2, t_1, t_3) \in \mathsf{Plus}$.