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- A two-place relation on sets S and T is called a *binary relation*. We often write "$s \; R \; t$" instead of $(s, t) \in R$. When $S = T$, R is a binary relation on S.

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- The *reflexive closure* of R, and the *reflexive and transitive closure* of R (R^*) are defined similarly.

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- Comma is used for *appending* sequences. If $a = 3, 1, 5$ and $b = 2, 6, 3, 1$, then:

$$a, b = 3, 1, 5, 2, 6, 3, 1$$

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$$a, 2 = 3, 1, 5, 2$$

2.4 Induction

In the following, P is a predicate on the natural numbers.

The *principle of ordinary induction on natural numbers* says that:

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Are the induction principles equivalent? Yes, and we will prove this on the blackboard.