CS 516—Software Foundations via Formal Languages—Spring 2022

Final Examination

Model Answers

Question 1

	union	concatenation	closure	intersection	difference
RegLan	Y	Y	Y	Y	Y
CFLan	Y	Y	Y	*N*	*N*
RecLan	Y	Y	Y	Y	Y
RELan	Y	Y	Y	Y	*N*

Question 2

$$\begin{aligned}
\mathsf{A} &\to \mathsf{B}\langle\mathsf{2}\rangle \mid \langle\mathsf{0}\rangle\mathsf{C}, \\
\mathsf{B} &\to \% \mid \mathsf{0B1}, \\
\mathsf{C} &\to \% \mid \mathsf{1C2}, \\
\langle\mathsf{0}\rangle &\to \% \mid \mathsf{0}\langle\mathsf{0}\rangle, \\
\langle\mathsf{2}\rangle &\to \% \mid \mathsf{2}\langle\mathsf{2}\rangle.
\end{aligned}$$

Question 3

To see that the statement is false, let $A = \{\mathsf{0}, \mathsf{10}\}$ and $B = \{\mathsf{01}, \mathsf{0}\}$. Then $\mathsf{010} \in A^*$ and $\mathsf{010} \in B^*$, so that $\mathsf{010} \in A^* \cap B^*$. But $A \cap B = \{\mathsf{0}\}$, so that $\mathsf{010} \notin (A \cap B)^*$. Thus $(A \cap B)^* \neq A^* \cap B^*$.

Question 4

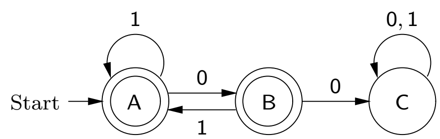
Because $x \notin Y$, it follows that $(Y \cup \{x\}) - \{x\} = Y$. We can start by defining a regular expression α_x such that $L(\alpha_x) = \{x\}$. (α_x will look just like x, in abbreviated form.) Then we can convert α_x into a DFA M_x, in a sequence of stages: regular expression to FA, FA to EFA, EFA to NFA, NFA to DFA, so that $L(M_x) = \{x\}$. Finally, we can define the DFA N by:

$$N = \textbf{minus}(M, M_x).$$

Then $L(N) = L(\textbf{minus}(M, M_x)) = L(M) - L(M_x) = (Y \cup \{x\}) - \{x\} = Y$.

Question 5

M is



First, we show by induction on Λ that:

(A) for all $w \in \Lambda_{\mathsf{A}}$, $w \in X$ and $\mathsf{0}$ is not a suffix of w;

(B) for all $w \in \Lambda_{\mathsf{B}}$, $w \in X$ and $\mathsf{0}$ is a suffix of w;

(C) for all $w \in \Lambda_{\mathsf{C}}$, $w \notin X$.

There are seven (one plus the number of transitions) parts to show.

(empty string) We must show that $\% \in X$ and $\mathsf{0}$ is not suffix of $\%$. The latter property is obvious, and the former follows by Lemma 5.1(1).

$(\mathsf{A}, \mathsf{0} \to \mathsf{B})$ Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis, $w \in X$ and $\mathsf{0}$ is not a suffix of w. We must show that $w\mathsf{0} \in X$ and $\mathsf{0}$ is a suffix of $w\mathsf{0}$. The latter property is obvious, and the former one follows by Lemma 5.1(3).

$(\mathsf{A}, \mathsf{1} \to \mathsf{A})$ Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis, $w \in X$ and $\mathsf{0}$ is not a suffix of w. We must show that $w\mathsf{1} \in X$ and $\mathsf{0}$ is not a suffix of $w\mathsf{1}$. The latter property is obvious, and the former one follows by Lemma 5.1(2).

$(\mathsf{B}, \mathsf{0} \to \mathsf{C})$ Suppose $w \in \Lambda_{\mathsf{B}}$, and assume the inductive hypothesis, $w \in X$ and $\mathsf{0}$ is a suffix of w. We must show that $w\mathsf{0} \notin X$, which follows by Lemma 5.1(4).

$(\mathsf{B}, \mathsf{1} \to \mathsf{A})$ Suppose $w \in \Lambda_{\mathsf{B}}$, and assume the inductive hypothesis, $w \in X$ and $\mathsf{0}$ is a suffix of w. We must show that $w\mathsf{1} \in X$ and $\mathsf{0}$ is not a suffix of $w\mathsf{1}$. The latter property is obvious, and the former one follows from Lemma 5.1(2).

$(\mathsf{C}, \mathsf{0} \to \mathsf{C})$ Suppose $w \in \Lambda_{\mathsf{C}}$, and assume the inductive hypothesis, $w \notin X$. We must show that $w\mathsf{0} \notin X$, and this follows by Lemma 5.1(5).

$(\mathsf{C}, \mathsf{1} \to \mathsf{C})$ Suppose $w \in \Lambda_{\mathsf{C}}$, and assume the inductive hypothesis, $w \notin X$. We must show that $w\mathsf{1} \notin X$, and this follows by Lemma 5.1(5).

Now we use the result of our induction on Λ to show that $L(M) = X$.

$(L(M) \subseteq X)$ Suppose $w \in L(M)$. Because $A_M = \{\mathsf{A}, \mathsf{B}\}$, we have that $w \in L(M) = \Lambda_{\mathsf{A}} \cup \Lambda_{\mathsf{B}}$. Thus by parts (A) and (B) of our induction on Λ, we have $w \in X$.

$(X \subseteq L(M))$ Suppose $w \in X$. Since $X \subseteq \{\mathsf{0}, \mathsf{1}\}^*$, we have that $w \in \{\mathsf{0}, \mathsf{1}\}^*$. Suppose, toward a contradiction, that $w \notin L(M)$. Because $w \notin L(M) = \Lambda_{\mathsf{A}} \cup \Lambda_{\mathsf{B}}$, and $w \in \{\mathsf{0}, \mathsf{1}\}^* = (\textbf{alphabet}\, M)^* = \Lambda_{\mathsf{A}} \cup \Lambda_{\mathsf{B}} \cup \Lambda_{\mathsf{C}}$, it follows that $w \in \Lambda_{\mathsf{C}}$. But part (C) of our induction on Λ tells us that $w \notin X$—contradiction. Thus $w \in L(M)$.

Question 6

Suppose, toward a contradiction, that L is regular. Thus there is an $n \in \mathbb{N} - \{0\}$ with the property of the Pumping Lemma for Regular Languages. Let $z = \mathsf{0}^{2n}\mathsf{1}^2\mathsf{2}^0\mathsf{3}^{2n+1}$. Because $2n < 2n+1$, $2 > 0$, $2n+2 = 2(n+1)$ is even, and $0 + (2n+1) = 2n+1$ is odd, we have that $z \in L$. Furthermore $|z| = 2n+2+0+(2n+1) = 4n+3 \geq n$, and thus the property of the lemma tells us that there are $u, v, w \in \textbf{Str}$ such that $z = uvw$ and

(1) $|uv| \leq n$; and

(2) $v \neq \%$; and

(3) $uv^i w \in L$, for all $i \in \mathbb{N}$.

Since $\mathsf{0}^{2n}\mathsf{1}^2\mathsf{2}^0\mathsf{3}^{2n+1} = z = uvw$, (1) tells us that $uv \in \{\mathsf{0}\}^*$. Thus (2) tells us that $v = \mathsf{0}^m$ for some $m \in \mathbb{N}$ such that $m \geq 1$. Thus $uv^2w = uvvw$ has exactly $2n + m$ $\mathsf{0}$'s, but has exactly $2n+1$ $\mathsf{3}$'s. Because $2n + m \geq 2n+1$, it follows that $uv^2w \notin L$. But according to (3), $uv^2w \in L$, giving us our contradiction. Thus L is not regular.