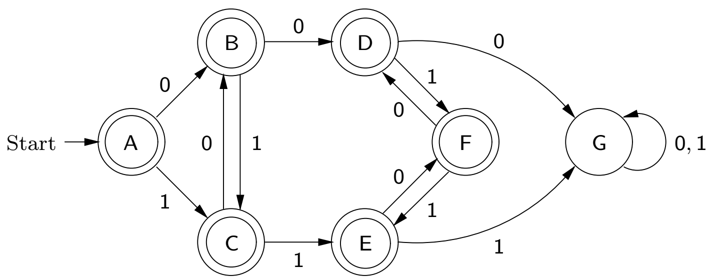
CS 591 S2—Formal Language Theory: Integrating Experimentation and Proof—Fall 2019

Problem Set 4

Model Answers

Problem 1

M is



Problem 2

First, we put the description of M in the file `ps4-dfa`:

```
{states} A, B, C, D, E, F, G
{start state} A
{accepting states} A, B, C, D, E, F
{transitions}
A, 0 -> B; A, 1 -> C;
B, 0 -> D; B, 1 -> C;
C, 0 -> B; C, 1 -> E;
D, 0 -> G; D, 1 -> F;
E, 0 -> F; E, 1 -> G;
F, 0 -> D; F, 1 -> E;
G, 0 -> G; G, 1 -> G
```

We then input M into Forlan, binding it to `dfa`:

```
- val dfa = DFA.input "ps4-dfa";
val dfa = - : dfa
```

Next, we put a testing harness in the file `ps4-p2.sml`:

```
(* val zero : sym
   val one  : sym

   the symbols 0 and 1 *)
```

```
val zero = Sym.fromString "0";
val one  = Sym.fromString "1";

(* val diff : str -> int

   the diff function, assuming input is a string over alphabet {0, 1}
   *)

fun diff (nil : str) = 0
  | diff (b :: bs)   =
      if Sym.equal(b, zero)
      then ~1 + diff bs
      else 1 + diff bs;

(* val allSubstringsGood : str -> bool

   tests whether all substrings of a string over alphabet {0, 1} have
   diff's between ~2 and 2, inclusive *)

fun allSubstringsGood x =
      Set.all
      (fn y =>
            let val n = diff y
            in ~2 <= n andalso n <= 2 end)
      (StrSet.substrings x);

(* val upto : int -> str set

   if n >= 0, then upto n returns all strings over alphabet {0, 1} of
   length no more than n *)

fun upto 0 : str set = Set.sing nil
  | upto n           =
      let val xs = upto(n - 1)
          val ys = Set.filter (fn x => length x = n - 1) xs
      in StrSet.union
         (xs, StrSet.concat(StrSet.fromString "0, 1", ys))
      end;

(* val partition : int -> str set * str set

   if n >= 0, then partition n returns (xs, ys) where:

   xs is all elements of upto n that are in X; and

   ys is all elements of upto n that are not in X *)

fun partition n = Set.partition allSubstringsGood (upto n);
```

```
(* val test = fn : int -> dfa -> str option * str option

   if n >= 0, then test n returns a function f such that, for all DFAs
   dfa, f dfa returns a pair (xOpt, yOpt) such that:

     If there is an element of {0, 1}* of length no more than n that
     is in X but is not accepted by dfa, then xOpt = SOME x for some
     such x; otherwise, xOpt = NONE.

     If there is an element of {0, 1}* of length no more than n that
     is not in X but is accepted by dfa, then yOpt = SOME y for some
     such y; otherwise, yOpt = NONE. *)

fun test n =
      let val (goods, bads) = partition n
      in fn dfa =>
              let val determAccepted = DFA.determAccepted dfa
                  val failAccOpt     = Set.position (not o determAccepted) goods
                  val failNotAccOpt  = Set.position determAccepted bads
              in ((case failAccOpt of
                        NONE   => NONE
                      | SOME i => SOME(ListAux.sub(Set.toList goods, i))),
                  (case failNotAccOpt of
                        NONE   => NONE
                      | SOME i => SOME(ListAux.sub(Set.toList bads, i))))
              end
      end;
```

We then load the testing harness into Forlan:

```
- use "ps4-p2.sml";
[opening ps4-p2.sml]
val zero = - : sym
val one = - : sym
val diff = fn : str -> int
val allSubstringsGood = fn : str -> bool
val upto = fn : int -> str set
val partition = fn : int -> str set * str set
val test = fn : int -> dfa -> str option * str option
val it = () : unit
```

Finally, we test M on all 65535 ($2^{16} - 1$) strings of length up to 15 as follows:

```
- val doit = test 15;
val doit = fn : dfa -> str option * str option
- doit dfa;
val it = (NONE,NONE) : str option * str option
```

Problem 3

Define a function **dsfxs** (for "diffs of suffixes") from $\{0,1\}^*$ to $\mathcal{P}\,\mathbb{Z}$ by: for all $w \in \{0,1\}^*$,

$$\textbf{dsfxs}\, w = \{\, \textbf{diff}\, v \mid v \text{ is a suffix of } w \,\}.$$

Given $Y \subseteq \mathbb{Z}$, we write

$$Y^+ = \{\, n+1 \mid n \in Y \,\}, \text{ and}$$
$$Y^- = \{\, n-1 \mid n \in Y \,\}.$$

Thus, we have that, for all $Y \subseteq \mathbb{Z}$:

- $n \in Y$ iff $n+1 \in Y^+$, for all $n \in \mathbb{Z}$;
- $n-1 \in Y$ iff $n \in Y^+$, for all $n \in \mathbb{Z}$;
- $n \in Y$ iff $n-1 \in Y^-$, for all $n \in \mathbb{Z}$; and
- $n+1 \in Y$ iff $n \in Y^-$, for all $n \in \mathbb{Z}$.

Lemma PS4.3.1

(1) $\% \in X$.

(2) For all $w \in \{0,1\}^*$, *if* $w \in X$ *and* $-2 \notin \textbf{dsfxs}\, w$, *then* $w0 \in X$.

(3) For all $w \in \{0,1\}^*$, *if* $w \in X$ *and* $2 \notin \textbf{dsfxs}\, w$, *then* $w1 \in X$.

Proof.

(1) Follows since $\%$ is the only substring of $\%$, $\textbf{diff}\, \% = 0$ and $-2 \leq 0 \leq 2$.

(2) Suppose $w \in \{0,1\}^*$, $w \in X$ and $-2 \notin \textbf{dsfxs}\, w$. To see that $w0 \in X$, suppose v is a substring of $w0$. We must show that $-2 \leq \textbf{diff}\, v \leq 2$. There are two cases to consider (if $v = \%$, then v is a substring of w).

- Suppose v is a substring of w. Then $-2 \leq \textbf{diff}\, v \leq 2$, since $w \in X$.
- Suppose $v = u0$, for a suffix u of w. Thus $\textbf{diff}\, v = \textbf{diff}(u0) = \textbf{diff}\, u + \textbf{diff}\, 0 = \textbf{diff}\, u + -1$. Because $w \in X$ and u is a substring of w, we have that $-2 \leq \textbf{diff}\, u \leq 2$. But $-2 \notin \textbf{dsfxs}\, w$, and so $\textbf{diff}\, u \neq -2$. Hence $-1 \leq \textbf{diff}\, u \leq 2$. Thus $-1 + -1 \leq \textbf{diff}\, u + -1 \leq 2 + -1$, so that $-2 = -1 + -1 \leq \textbf{diff}\, v \leq 2 + -1 = 1 \leq 2$.

(3) Suppose $w \in \{0,1\}^*$, $w \in X$ and $2 \notin \textbf{dsfxs}\, w$. To see that $w1 \in X$, suppose v is a substring of $w1$. We must show that $-2 \leq \textbf{diff}\, v \leq 2$. There are two cases to consider (if $v = \%$, then v is a substring of w).

- Suppose v is a substring of w. Then $-2 \leq \textbf{diff}\, v \leq 2$, since $w \in X$.
- Suppose $v = u1$, for a suffix u of w. Thus $\textbf{diff}\, v = \textbf{diff}(u1) = \textbf{diff}\, u + \textbf{diff}\, 1 = \textbf{diff}\, u + 1$. Because $w \in X$ and u is a substring of w, we have that $-2 \leq \textbf{diff}\, u \leq 2$. But $2 \notin \textbf{dsfxs}\, w$, and so $\textbf{diff}\, u \neq 2$. Hence $-2 \leq \textbf{diff}\, u \leq 1$. Thus $-2 + 1 \leq \textbf{diff}\, u + 1 \leq 1 + 1$, so that $-2 \leq -1 = -2 + 1 \leq \textbf{diff}\, v \leq 1 + 1 = 2$.

□

Lemma PS4.3.2

(1) $\mathbf{dsfxs}\,\% = \{0\}$.

(2) For all $w \in \{0,1\}^*$, $\mathbf{dsfxs}(w0) = (\mathbf{dsfxs}\,w)^- \cup \{0\}$.

(3) For all $w \in \{0,1\}^*$, $\mathbf{dsfxs}(w1) = (\mathbf{dsfxs}\,w)^+ \cup \{0\}$.

Proof.

(1) We show that $\mathbf{dsfxs}\,\% \subseteq \{0\} \subseteq \mathbf{dsfxs}\,\%$.

($\mathbf{dsfxs}\,\% \subseteq \{0\}$) Suppose $n \in \mathbf{dsfxs}\,\%$. Then $n = \mathbf{diff}\,v$ for some suffix v of %. Hence $v = \%$, so that $n = \mathbf{diff}\,v = \mathbf{diff}\,\% = 0 \in \{0\}$.

($\{0\} \subseteq \mathbf{dsfxs}\,\%$) Since % is a suffix of %, we have that $0 = \mathbf{diff}\,\% \in \mathbf{dsfxs}\,\%$. Hence $\{0\} \subseteq \mathbf{dsfxs}\,\%$.

(2) Suppose $w \in \{0,1\}^*$. We show that $\mathbf{dsfxs}(w0) \subseteq (\mathbf{dsfxs}\,w)^- \cup \{0\} \subseteq \mathbf{dsfxs}(w0)$.

($\mathbf{dsfxs}(w0) \subseteq (\mathbf{dsfxs}\,w)^- \cup \{0\}$) Suppose $n \in \mathbf{dsfxs}(w0)$. Hence $n = \mathbf{diff}\,v$ for some suffix v of $w0$. There are two cases to consider.

- Suppose $v = \%$. Then $n = \mathbf{diff}\,v = \mathbf{diff}\,\% = 0 \in \{0\} \subseteq (\mathbf{dsfxs}\,w)^- \cup \{0\}$.
- Suppose $v = u0$, for some suffix u of w. Thus $n = \mathbf{diff}\,v = \mathbf{diff}(u0) = \mathbf{diff}\,u + \mathbf{diff}\,0 = \mathbf{diff}\,u + -1 = \mathbf{diff}\,u - 1$. Since u is a suffix of w, we have that $\mathbf{diff}\,u \in \mathbf{dsfxs}\,w$, so that $\mathbf{diff}\,u - 1 \in (\mathbf{dsfxs}\,w)^-$. Hence $n = \mathbf{diff}\,u - 1 \in (\mathbf{dsfxs}\,w)^- \subseteq (\mathbf{dsfxs}\,w)^- \cup \{0\}$.

($(\mathbf{dsfxs}\,w)^- \cup \{0\} \subseteq \mathbf{dsfxs}(w0)$) Suppose $n \in (\mathbf{dsfxs}\,w)^- \cup \{0\}$. There are two cases to consider.

- Suppose $n \in (\mathbf{dsfxs}\,w)^-$. Thus $n+1 \in \mathbf{dsfxs}\,w$, so that $n+1 = \mathbf{diff}\,v$, for some suffix v of w. Thus $v0$ is a suffix of $w0$, so that $n = \mathbf{diff}\,v - 1 = \mathbf{diff}\,v + -1 = \mathbf{diff}\,v + \mathbf{diff}\,0 = \mathbf{diff}(v0) \in \mathbf{dsfxs}(w0)$.
- Suppose $n \in \{0\}$. Since % is a suffix of $w0$, we have that $n = 0 = \mathbf{diff}\,\% \in \mathbf{dsfxs}(w0)$.

(3) Suppose $w \in \{0,1\}^*$. We show that $\mathbf{dsfxs}(w1) \subseteq (\mathbf{dsfxs}\,w)^+ \cup \{0\} \subseteq \mathbf{dsfxs}(w1)$.

($\mathbf{dsfxs}(w1) \subseteq (\mathbf{dsfxs}\,w)^+ \cup \{0\}$) Suppose $n \in \mathbf{dsfxs}(w1)$. Hence $n = \mathbf{diff}\,v$ for some suffix v of $w1$. There are two cases to consider.

- Suppose $v = \%$. Then $n = \mathbf{diff}\,v = \mathbf{diff}\,\% = 0 \in \{0\} \subseteq (\mathbf{dsfxs}\,w)^+ \cup \{0\}$.
- Suppose $v = u1$, for some suffix u of w. Thus $n = \mathbf{diff}\,v = \mathbf{diff}(u1) = \mathbf{diff}\,u + \mathbf{diff}\,1 = \mathbf{diff}\,u + 1$. Since u is a suffix of w, we have that $\mathbf{diff}\,u \in \mathbf{dsfxs}\,w$, so that $\mathbf{diff}\,u + 1 \in (\mathbf{dsfxs}\,w)^+$. Hence $n = \mathbf{diff}\,u + 1 \in (\mathbf{dsfxs}\,w)^+ \subseteq (\mathbf{dsfxs}\,w)^+ \cup \{0\}$.

($(\mathbf{dsfxs}\,w)^+ \cup \{0\} \subseteq \mathbf{dsfxs}(w1)$) Suppose $n \in (\mathbf{dsfxs}\,w)^+ \cup \{0\}$. There are two cases to consider.

- Suppose $n \in (\mathbf{dsfxs}\,w)^+$. Thus $n-1 \in \mathbf{dsfxs}\,w$, so that $n-1 = \mathbf{diff}\,v$, for some suffix v of w. Thus $v1$ is a suffix of $w1$, so that $n = \mathbf{diff}\,v + 1 = \mathbf{diff}\,v + \mathbf{diff}\,1 = \mathbf{diff}(v1) \in \mathbf{dsfxs}(w1)$.

- Suppose $n \in \{0\}$. Since $\%$ is a suffix of $w1$, we have that $n = 0 = \textbf{diff}\,\% \in \textbf{dsfxs}(w1)$.

□

Lemma PS4.3.3

(A) For all $w \in \Lambda_{\mathsf{A}}$, $w \in X$ and $\textbf{dsfxs}\,w = \{0\}$.

(B) For all $w \in \Lambda_{\mathsf{B}}$, $w \in X$ and $\textbf{dsfxs}\,w = \{-1, 0\}$.

(C) For all $w \in \Lambda_{\mathsf{C}}$, $w \in X$ and $\textbf{dsfxs}\,w = \{0, 1\}$.

(D) For all $w \in \Lambda_{\mathsf{D}}$, $w \in X$ and $\textbf{dsfxs}\,w = \{-2, -1, 0\}$.

(E) For all $w \in \Lambda_{\mathsf{E}}$, $w \in X$ and $\textbf{dsfxs}\,w = \{0, 1, 2\}$.

(F) For all $w \in \Lambda_{\mathsf{F}}$, $w \in X$ and $\textbf{dsfxs}\,w = \{-1, 0, 1\}$.

(G) For all $w \in \Lambda_{\mathsf{G}}$, $w \notin X$.

Proof. We proceed by induction on Λ. There are 15 (1 plus the number of transitions) parts to show.

(empty string) We must show that $\% \in X$ and $\textbf{dsfxs}\,\% = \{0\}$, which follows by Lemma PS4.3.1(1) and Lemma PS4.3.2(1).

($\mathsf{A}, \mathsf{0} \to \mathsf{B}$) Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w \in X$ and $\textbf{dsfxs}\,w = \{0\}$. We must show that $w\mathsf{0} \in X$ and $\textbf{dsfxs}(w\mathsf{0}) = \{-1, 0\}$. Since $w \in X$ and $-2 \notin \textbf{dsfxs}\,w$, Lemma PS4.3.1(2) tells us that $w\mathsf{0} \in X$. And, by Lemma PS4.3.2(2), we have that $\textbf{dsfxs}(w\mathsf{0}) = (\textbf{dsfxs}\,w)^- \cup \{0\} = \{0\}^- \cup \{0\} = \{-1\} \cup \{0\} = \{-1, 0\}$.

($\mathsf{A}, \mathsf{1} \to \mathsf{C}$) Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w \in X$ and $\textbf{dsfxs}\,w = \{0\}$. We must show that $w\mathsf{1} \in X$ and $\textbf{dsfxs}(w\mathsf{1}) = \{0, 1\}$. Since $w \in X$ and $2 \notin \textbf{dsfxs}\,w$, Lemma PS4.3.1(3) tells us that $w\mathsf{1} \in X$. And, by Lemma PS4.3.2(3), we have that $\textbf{dsfxs}(w\mathsf{1}) = (\textbf{dsfxs}\,w)^+ \cup \{0\} = \{0\}^+ \cup \{0\} = \{1\} \cup \{0\} = \{0, 1\}$.

($\mathsf{B}, \mathsf{0} \to \mathsf{D}$) Suppose $w \in \Lambda_{\mathsf{B}}$, and assume the inductive hypothesis: $w \in X$ and $\textbf{dsfxs}\,w = \{-1, 0\}$. We must show that $w\mathsf{0} \in X$ and $\textbf{dsfxs}(w\mathsf{0}) = \{-2, -1, 0\}$. Since $w \in X$ and $-2 \notin \textbf{dsfxs}\,w$, Lemma PS4.3.1(2) tells us that $w\mathsf{0} \in X$. And, by Lemma PS4.3.2(2), we have that $\textbf{dsfxs}(w\mathsf{0}) = (\textbf{dsfxs}\,w)^- \cup \{0\} = \{-1, 0\}^- \cup \{0\} = \{-2, -1\} \cup \{0\} = \{-2, -1, 0\}$.

($\mathsf{B}, \mathsf{1} \to \mathsf{C}$) Suppose $w \in \Lambda_{\mathsf{B}}$, and assume the inductive hypothesis: $w \in X$ and $\textbf{dsfxs}\,w = \{-1, 0\}$. We must show that $w\mathsf{1} \in X$ and $\textbf{dsfxs}(w\mathsf{1}) = \{0, 1\}$. Since $w \in X$ and $2 \notin \textbf{dsfxs}\,w$, Lemma PS4.3.1(3) tells us that $w\mathsf{1} \in X$. And, by Lemma PS4.3.2(3), we have that $\textbf{dsfxs}(w\mathsf{1}) = (\textbf{dsfxs}\,w)^+ \cup \{0\} = \{-1, 0\}^+ \cup \{0\} = \{0, 1\} \cup \{0\} = \{0, 1\}$.

($\mathsf{C}, \mathsf{0} \to \mathsf{B}$) Suppose $w \in \Lambda_{\mathsf{C}}$, and assume the inductive hypothesis: $w \in X$ and $\textbf{dsfxs}\,w = \{0, 1\}$. We must show that $w\mathsf{0} \in X$ and $\textbf{dsfxs}(w\mathsf{0}) = \{-1, 0\}$. Since $w \in X$ and $-2 \notin \textbf{dsfxs}\,w$, Lemma PS4.3.1(2) tells us that $w\mathsf{0} \in X$. And, by Lemma PS4.3.2(2), we have that $\textbf{dsfxs}(w\mathsf{0}) = (\textbf{dsfxs}\,w)^- \cup \{0\} = \{0, 1\}^- \cup \{0\} = \{-1, 0\} \cup \{0\} = \{-1, 0\}$.

($\mathsf{C}, \mathsf{1} \to \mathsf{E}$) Suppose $w \in \Lambda_{\mathsf{C}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{0, 1\}$. We must show that $w\mathsf{1} \in X$ and $\mathbf{dsfxs}(w\mathsf{1}) = \{0, 1, 2\}$. Since $w \in X$ and $2 \notin \mathbf{dsfxs}\, w$, Lemma PS4.3.1(3) tells us that $w\mathsf{1} \in X$. And, by Lemma PS4.3.2(3), we have that $\mathbf{dsfxs}(w\mathsf{1}) = (\mathbf{dsfxs}\, w)^+ \cup \{0\} = \{0, 1\}^+ \cup \{0\} = \{1, 2\} \cup \{0\} = \{0, 1, 2\}$.

($\mathsf{D}, \mathsf{0} \to \mathsf{G}$) Suppose $w \in \Lambda_{\mathsf{D}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{-2, -1, 0\}$. We must show that $w\mathsf{0} \notin X$. Since $-2 \in \mathbf{dsfxs}\, w$, there is a suffix v of w such that $\mathbf{diff}\, v = -2$. Thus $v\mathsf{0}$ is a suffix of $w\mathsf{0}$, and $\mathbf{diff}(v\mathsf{0}) = \mathbf{diff}\, v + \mathbf{diff}\, \mathsf{0} = -2 + -1 = -3$. Since $v\mathsf{0}$ is a substring of $w\mathsf{0}$ and $\mathbf{diff}(v\mathsf{0}) = -3$, it follows that $w\mathsf{0} \notin X$.

($\mathsf{D}, \mathsf{1} \to \mathsf{F}$) Suppose $w \in \Lambda_{\mathsf{D}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{-2, -1, 0\}$. We must show that $w\mathsf{1} \in X$ and $\mathbf{dsfxs}(w\mathsf{1}) = \{-1, 0, 1\}$. Since $w \in X$ and $2 \notin \mathbf{dsfxs}\, w$, Lemma PS4.3.1(3) tells us that $w\mathsf{1} \in X$. And, by Lemma PS4.3.2(3), we have that $\mathbf{dsfxs}(w\mathsf{1}) = (\mathbf{dsfxs}\, w)^+ \cup \{0\} = \{-2, -1, 0\}^+ \cup \{0\} = \{-1, 0, 1\} \cup \{0\} = \{-1, 0, 1\}$.

($\mathsf{E}, \mathsf{0} \to \mathsf{F}$) Suppose $w \in \Lambda_{\mathsf{E}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{0, 1, 2\}$. We must show that $w\mathsf{0} \in X$ and $\mathbf{dsfxs}(w\mathsf{0}) = \{-1, 0, 1\}$. Since $w \in X$ and $-2 \notin \mathbf{dsfxs}\, w$, Lemma PS4.3.1(2) tells us that $w\mathsf{0} \in X$. And, by Lemma PS4.3.2(2), we have that $\mathbf{dsfxs}(w\mathsf{0}) = (\mathbf{dsfxs}\, w)^- \cup \{0\} = \{0, 1, 2\}^- \cup \{0\} = \{-1, 0, 1\} \cup \{0\} = \{-1, 0, 1\}$.

($\mathsf{E}, \mathsf{1} \to \mathsf{G}$) Suppose $w \in \Lambda_{\mathsf{E}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{0, 1, 2\}$. We must show that $w\mathsf{1} \notin X$. Since $2 \in \mathbf{dsfxs}\, w$, there is a suffix v of w such that $\mathbf{diff}\, v = 2$. Thus $v\mathsf{1}$ is a suffix of $w\mathsf{1}$, and $\mathbf{diff}(v\mathsf{1}) = \mathbf{diff}\, v + \mathbf{diff}\, \mathsf{1} = 2 + 1 = 3$. Since $v\mathsf{1}$ is a substring of $w\mathsf{1}$ and $\mathbf{diff}(v\mathsf{1}) = 3$, it follows that $w\mathsf{1} \notin X$.

($\mathsf{F}, \mathsf{0} \to \mathsf{D}$) Suppose $w \in \Lambda_{\mathsf{F}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{-1, 0, 1\}$. We must show that $w\mathsf{0} \in X$ and $\mathbf{dsfxs}(w\mathsf{0}) = \{-2, -1, 0\}$. Since $w \in X$ and $-2 \notin \mathbf{dsfxs}\, w$, Lemma PS4.3.1(2) tells us that $w\mathsf{0} \in X$. And, by Lemma PS4.3.2(2), we have that $\mathbf{dsfxs}(w\mathsf{0}) = (\mathbf{dsfxs}\, w)^- \cup \{0\} = \{-1, 0, 1\}^- \cup \{0\} = \{-2, -1, 0\} \cup \{0\} = \{-2, -1, 0\}$.

($\mathsf{F}, \mathsf{1} \to \mathsf{E}$) Suppose $w \in \Lambda_{\mathsf{F}}$, and assume the inductive hypothesis: $w \in X$ and $\mathbf{dsfxs}\, w = \{-1, 0, 1\}$. We must show that $w\mathsf{1} \in X$ and $\mathbf{dsfxs}(w\mathsf{1}) = \{0, 1, 2\}$. Since $w \in X$ and $2 \notin \mathbf{dsfxs}\, w$, Lemma PS4.3.1(3) tells us that $w\mathsf{1} \in X$. And, by Lemma PS4.3.2(3), we have that $\mathbf{dsfxs}(w\mathsf{1}) = (\mathbf{dsfxs}\, w)^+ \cup \{0\} = \{-1, 0, 1\}^+ \cup \{0\} = \{0, 1, 2\} \cup \{0\} = \{0, 1, 2\}$.

($\mathsf{G}, \mathsf{0} \to \mathsf{G}$) Suppose $w \in \Lambda_{\mathsf{G}}$, and assume the inductive hypothesis: $w \notin X$. We must show that $w\mathsf{0} \notin X$. Since $w \notin X$ but $w \in \Lambda_{\mathsf{G}} \subseteq (\mathbf{alphabet}\, M)^* = \{\mathsf{0}, \mathsf{1}\}^*$, it follows that there is a substring v of w such $-2 \leq \mathbf{diff}\, v \leq 2$ is false. But v is also a substring of $w\mathsf{0}$, and so $w\mathsf{0} \notin X$.

($\mathsf{G}, \mathsf{1} \to \mathsf{G}$) Suppose $w \in \Lambda_{\mathsf{G}}$, and assume the inductive hypothesis: $w \notin X$. We must show that $w\mathsf{1} \notin X$. Since $w \notin X$ but $w \in \Lambda_{\mathsf{G}} \subseteq (\mathbf{alphabet}\, M)^* = \{\mathsf{0}, \mathsf{1}\}^*$, it follows that there is a substring v of w such $-2 \leq \mathbf{diff}\, v \leq 2$ is false. But v is also a substring of $w\mathsf{1}$, and so $w\mathsf{1} \notin X$.

□

Proposition PS4.3.4
$L(M) = X$.

Proof. We show that $L(M) \subseteq X \subseteq L(M)$.

$(L(M) \subseteq X)$ Suppose $w \in L(M)$. Because $A_M = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}\}$, we have that $w \in L(M) = \Lambda_{\mathsf{A}} \cup \Lambda_{\mathsf{B}} \cup \Lambda_{\mathsf{C}} \cup \Lambda_{\mathsf{D}} \cup \Lambda_{\mathsf{E}} \cup \Lambda_{\mathsf{F}}$. Thus, by Lemma PS4.3.3(A–F), we have that $w \in X$.

$(X \subseteq L(M))$ Suppose $w \in X$. Since $X \subseteq \{\mathsf{0}, \mathsf{1}\}^*$, we have that $w \in \{\mathsf{0}, \mathsf{1}\}^*$. Suppose, toward a contradiction, that $w \notin L(M)$. Because $w \notin L(M) = \Lambda_{\mathsf{A}} \cup \Lambda_{\mathsf{B}} \cup \Lambda_{\mathsf{C}} \cup \Lambda_{\mathsf{D}} \cup \Lambda_{\mathsf{E}} \cup \Lambda_{\mathsf{F}}$ and $w \in \{\mathsf{0}, \mathsf{1}\}^* = (\mathbf{alphabet}\, M)^* = \Lambda_{\mathsf{A}} \cup \Lambda_{\mathsf{B}} \cup \Lambda_{\mathsf{C}} \cup \Lambda_{\mathsf{D}} \cup \Lambda_{\mathsf{E}} \cup \Lambda_{\mathsf{F}} \cup \Lambda_{\mathsf{G}}$, we must have that $w \in \Lambda_{\mathsf{G}}$. But then Lemma PS4.3.3(G) tells us that $w \notin X$—contradiction. Thus $w \in L(M)$.

□