CS 591 S2—Formal Language Theory: Integrating Experimentation and Proof—Fall 2019

Problem Set 7

Model Answers

Problem 1

Let

$$Y = \{\, 1^j 2^k \mid j, k \in \mathbb{N} \text{ and } j > k \text{ and } j \text{ is odd and } k \text{ is even} \,\},$$
$$Z = \{\, 3^l \mid l \in \mathbb{N} \text{ and } l \text{ is odd} \,\}.$$

Lemma PS7.1.1

(A) For all $w \in \Pi_{\mathsf{A}}$, $w \in X$.

(B) For all $w \in \Pi_{\mathsf{B}}$, $w \in Y$.

(C) For all $w \in \Pi_{\mathsf{C}}$, $w \in Z$.

Proof. By induction on Π. There are ten productions to consider.

($\mathsf{A} \to \mathsf{1BC}$) Suppose $y \in \Pi_{\mathsf{B}}$ and $z \in \Pi_{\mathsf{C}}$, and assume the inductive hypothesis: $y \in Y$ and $z \in Z$. We must show that $\mathsf{1}yz \in X$. We have that: $y = \mathsf{1}^j\mathsf{2}^k$ for some $j, k \in \mathbb{N}$ such that $j > k$, j is odd, and k is even; and $z = \mathsf{3}^l$ for some $l \in \mathbb{N}$ such that l is odd. Thus $\mathsf{1}yz = \mathsf{0}^0\mathsf{1}^{j+1}\mathsf{2}^k\mathsf{3}^l$. Since l is odd, we have $0 < l$. Because $j > k$, we have $j + 1 > k$. Since j is odd, we have $0 + (j + 1)$ is even. Because k is even and l is odd, we have $k + l$ is odd. Thus $\mathsf{1}yz = \mathsf{0}^0\mathsf{1}^{j+1}\mathsf{2}^k\mathsf{3}^l \in X$.

($\mathsf{A} \to \mathsf{1B23C}$) Suppose $y \in \Pi_{\mathsf{B}}$ and $z \in \Pi_{\mathsf{C}}$, and assume the inductive hypothesis: $y \in Y$ and $z \in Z$. We must show that $\mathsf{1}y\mathsf{23}z \in X$. We have that: $y = \mathsf{1}^j\mathsf{2}^k$ for some $j, k \in \mathbb{N}$ such that $j > k$, j is odd, and k is even; and $z = \mathsf{3}^l$ for some $l \in \mathbb{N}$ such that l is odd. Thus $\mathsf{1}y\mathsf{23}z = \mathsf{0}^0\mathsf{1}^{j+1}\mathsf{2}^{k+1}\mathsf{3}^{l+1}$. Clearly, $0 < l+1$. Since $j > k$, we have $j+1 > k+1$. Since j is odd, we have $0+(j+1)$ is even. Because k is even and l is odd, we have $(k+1)+(l+1) = k+l+2$ is odd. Thus $\mathsf{1}y\mathsf{23}z = \mathsf{0}^0\mathsf{1}^{j+1}\mathsf{2}^{k+1}\mathsf{3}^{l+1} \in X$.

($\mathsf{A} \to \mathsf{0B33C}$) Suppose $y \in \Pi_{\mathsf{B}}$ and $z \in \Pi_{\mathsf{C}}$, and assume the inductive hypothesis: $y \in Y$ and $z \in Z$. We must show that $\mathsf{0}y\mathsf{33}z \in X$. We have that: $y = \mathsf{1}^j\mathsf{2}^k$ for some $j, k \in \mathbb{N}$ such that $j > k$, j is odd, and k is even; and $z = \mathsf{3}^l$ for some $l \in \mathbb{N}$ such that l is odd. Thus $\mathsf{0}y\mathsf{33}z = \mathsf{0}^1\mathsf{1}^j\mathsf{2}^k\mathsf{3}^{l+2}$. Clearly, $1 < l + 2$ and $j > k$. Because j is odd, we have $1 + j$ is even. Since k is even and l is odd, we have $k + l + 2$ is odd. Thus $\mathsf{0}y\mathsf{33}z = \mathsf{0}^1\mathsf{1}^j\mathsf{2}^k\mathsf{3}^{l+2} \in X$.

($\mathsf{A} \to \mathsf{011B23C}$) Suppose $y \in \Pi_{\mathsf{B}}$ and $z \in \Pi_{\mathsf{C}}$, and assume the inductive hypothesis: $y \in Y$ and $z \in Z$. We must show that $\mathsf{011}y\mathsf{23}z \in X$. We have that: $y = \mathsf{1}^j\mathsf{2}^k$ for some $j, k \in \mathbb{N}$ such that $j > k$, j is odd, and k is even; and $z = \mathsf{3}^l$ for some $l \in \mathbb{N}$ such that l is odd. Thus $\mathsf{011}y\mathsf{23}z = \mathsf{0}^1\mathsf{1}^{j+2}\mathsf{2}^{k+1}\mathsf{3}^{l+1}$. Since l is odd, we have $1 < l + 1$. Since $j > k$, we have $j + 2 > k + 1$. Since j is odd, we have $1 + (j + 2) = j + 3$ is even. Since k is even and l is odd, we have $(k + 1) + (l + 1) = k + l + 2$ is odd. Thus $\mathsf{011}y\mathsf{23}z = \mathsf{0}^1\mathsf{1}^{j+2}\mathsf{2}^{k+1}\mathsf{3}^{l+1} \in X$.

($\mathsf{A} \to \mathsf{00A33}$) Suppose $x \in \Pi_\mathsf{A}$, and assume the inductive hypothesis: $x \in X$. We must show that $\mathsf{00}x\mathsf{33} \in X$. We have that $x = \mathsf{0}^i\mathsf{1}^j\mathsf{2}^k\mathsf{3}^l$, for some $i, j, k, l \in \mathbb{N}$ such that $i < l$, $j > k$, $i + j$ is even, and $k + l$ is odd. Thus $\mathsf{00}x\mathsf{33} = \mathsf{0}^{i+2}\mathsf{1}^j\mathsf{2}^k\mathsf{3}^{l+2}$. Since $i < l$, we have $i + 2 < l + 2$. Clearly, $j > k$. Since $i + j$ is even, we have $i + 2 + j$ is even. Because $k + l$ is odd, we have $k + l + 2$ is odd. Thus $\mathsf{00}x\mathsf{33} = \mathsf{0}^{i+2}\mathsf{1}^j\mathsf{2}^k\mathsf{3}^{l+2} \in X$.

($\mathsf{B} \to \mathsf{1}$) We must show that $\mathsf{1} \in Y$. We have that $\mathsf{1} = \mathsf{1}^1\mathsf{2}^0$. Because $1 > 0$ and 1 is odd and 0 is even, we have $\mathsf{1} = \mathsf{1}^1\mathsf{2}^0 \in Y$.

($\mathsf{B} \to \mathsf{11B}$) Suppose $y \in \Pi_\mathsf{B}$, and assume the inductive hypothesis: $y \in Y$. We must show that $\mathsf{11}y \in Y$. We have that $y = \mathsf{1}^j\mathsf{2}^k$ for some $j, k \in \mathbb{N}$ such that $j > k$, j is odd, and k is even. Thus $\mathsf{11}y = \mathsf{1}^{j+2}\mathsf{2}^k$. Since $j > k$, we have $j + 2 > k$. Because j is odd, we have $j + 2$ is odd. Clearly, k is even. Thus $\mathsf{11}y = \mathsf{1}^{j+2}\mathsf{2}^k \in Y$.

($\mathsf{B} \to \mathsf{11B22}$) Suppose $y \in \Pi_\mathsf{B}$, and assume the inductive hypothesis: $y \in Y$. We must show that $\mathsf{11}y\mathsf{22} \in Y$. We have that $y = \mathsf{1}^j\mathsf{2}^k$ for some $j, k \in \mathbb{N}$ such that $j > k$, j is odd, and k is even. Thus $\mathsf{11}y\mathsf{22} = \mathsf{1}^{j+2}\mathsf{2}^{k+2}$. Since $j > k$, we have $j + 2 > k + 2$. Because j is odd, we have $j + 2$ is odd. Since k is even, we have $k + 2$ is even. Thus $\mathsf{11}y\mathsf{22} = \mathsf{1}^{j+2}\mathsf{2}^{k+2} \in Y$.

($\mathsf{C} \to \mathsf{3}$) We must show that $\mathsf{3} \in Z$. We have $\mathsf{3} = \mathsf{3}^1$ and 1 is odd. Thus $\mathsf{3} = \mathsf{3}^1 \in Z$.

($\mathsf{C} \to \mathsf{33C}$) Suppose $z \in \Pi_\mathsf{C}$, and assume the inductive hypothesis: $z \in Z$. We must show that $\mathsf{33}z \in Z$. We have that $z = \mathsf{3}^l$ for some $l \in \mathbb{N}$ such that l is odd. Thus $\mathsf{33}z = \mathsf{3}^{l+2}$. Since l is odd, we have $l + 2$ is odd. Thus $\mathsf{33}z = \mathsf{3}^{l+2} \in Z$.

□

Lemma PS7.1.2
For all $z \in Z$, $z \in \Pi_\mathsf{C}$.

Proof. Suppose $z \in Z$, so that $z = \mathsf{3}^l$ for some $l \in \mathbb{N}$ such that l is odd. We must show that $z \in \Pi_\mathsf{C}$. It will suffice to show that, for all $l \in \mathbb{N}$,

if l is odd, then $\mathsf{3}^l \in \Pi_\mathsf{C}$.

We proceed by strong induction. Suppose $l \in \mathbb{N}$, and assume the inductive hypothesis: for all $l' \in \mathbb{N}$, if $l' < l$, then,

if l' is odd, then $\mathsf{3}^{l'} \in \Pi_\mathsf{C}$.

We must show that,

if l is odd, then $\mathsf{3}^l \in \Pi_\mathsf{C}$.

Suppose l is odd. We must show that $\mathsf{3}^l \in \Pi_\mathsf{C}$. There are two cases to consider.

($l = 1$) Then $\mathsf{3}^l = \mathsf{3} \in \Pi_\mathsf{C}$, because of the production $\mathsf{C} \to \mathsf{3}$ of G.

($l \geq 3$) Then $l - 2 \in \mathbb{N}$, $l - 2 < l$ and $l - 2$ is odd, so that, by the inductive hypothesis, $\mathsf{3}^{l-2} \in \Pi_\mathsf{C}$. Thus $\mathsf{3}^l = \mathsf{33}\mathsf{3}^{l-2} \in \Pi_\mathsf{C}$, because of the production $\mathsf{C} \to \mathsf{33C}$.

□

Lemma PS7.1.3
For all $y \in Y$, $y \in \Pi_{\mathsf{B}}$.

Proof. Suppose $y \in Y$, so that $y = \mathsf{1}^j\mathsf{2}^k$ for some $j, k \in \mathbb{N}$ such that $j > k$, j is odd, and k is even. We must show that $y \in \Pi_{\mathsf{B}}$. It will suffice to show that, for all $j \in \mathbb{N}$,

for all $k \in \mathbb{N}$, if $j > k$, j is odd, and k is even, then $\mathsf{1}^j\mathsf{2}^k \in \Pi_{\mathsf{B}}$.

We proceed by strong induction. Suppose $j \in \mathbb{N}$, and assume the inductive hypothesis: for all $j' \in \mathbb{N}$, if $j' < j$, then,

for all $k \in \mathbb{N}$, if $j' > k$, j' is odd, and k is even, then $\mathsf{1}^{j'}\mathsf{2}^k \in \Pi_{\mathsf{B}}$.

We must show that,

for all $k \in \mathbb{N}$, if $j > k$, j is odd, and k is even, then $\mathsf{1}^j\mathsf{2}^k \in \Pi_{\mathsf{B}}$.

Suppose $k \in \mathbb{N}$, $j > k$, j is odd, and k is even. We must show that $\mathsf{1}^j\mathsf{2}^k \in \Pi_{\mathsf{B}}$. Because j is odd, there are two cases to consider.

$(j = 1)$ Because $j > k$, it follows that $k = 0$. Then $\mathsf{1}^j\mathsf{2}^k = \mathsf{1} \in \Pi_{\mathsf{B}}$, because of the production $\mathsf{B} \to \mathsf{1}$ of G.

$(j \geq 3)$ Then $j - 2 \in \mathbb{N}$ and $j - 2$ is odd. Because k is even, there are two subcases to consider.

- $(k = 0)$ Because $j-2$ is odd, we have that $j-2 > k$. Since $j-2 \in \mathbb{N}$, $j-2 < j$, $k \in \mathbb{N}$, $j-2 > k$, $j-2$ is odd, and k is even, the inductive hypothesis tells us that $\mathsf{1}^{j-2} = \mathsf{1}^{j-2}\mathsf{2}^k \in \Pi_{\mathsf{B}}$. Thus $\mathsf{1}^j\mathsf{2}^k = \mathsf{11}\mathsf{1}^{j-2} \in \Pi_{\mathsf{B}}$, because of the production $\mathsf{B} \to \mathsf{11B}$ of G.
- $(k \geq 2)$ Because $j > k \geq 2$, we have that $j - 2 > k - 2$, $k - 2 \in \mathbb{N}$ and $k - 2$ is even. Since $j-2 \in \mathbb{N}$, $j-2 < j$, $k-2 \in \mathbb{N}$, $j-2 > k-2$, $j-2$ is odd, and $k-2$ is even, the inductive hypothesis tells us that $\mathsf{1}^{j-2}\mathsf{2}^{k-2} \in \Pi_{\mathsf{B}}$. Thus $\mathsf{1}^j\mathsf{2}^k = \mathsf{11}(\mathsf{1}^{j-2}\mathsf{2}^{k-2})\mathsf{22} \in \Pi_{\mathsf{B}}$, because of the production $\mathsf{B} \to \mathsf{11B22}$ of G.

□

Lemma PS7.1.4
For all $x \in X$, $x \in \Pi_{\mathsf{A}}$.

Proof. Suppose $x \in X$, so that $x = \mathsf{0}^i\mathsf{1}^j\mathsf{2}^k\mathsf{3}^l$ for some $i, j, k, l \in \mathbb{N}$ such that $i < l$, $j > k$, $i + j$ is even, and $k + l$ is odd. We must show that $x \in \Pi_{\mathsf{A}}$. It will suffice to show that, for all $i \in \mathbb{N}$,

for all $j, k, l \in \mathbb{N}$, if $i < l$, $j > k$, $i + j$ is even, and $k + l$ is odd, then $\mathsf{0}^i\mathsf{1}^j\mathsf{2}^k\mathsf{3}^l \in \Pi_{\mathsf{A}}$.

We proceed by strong induction. Suppose $i \in \mathbb{N}$, and assume the inductive hypothesis: for all $i' \in \mathbb{N}$, if $i' < j$, then,

for all $j, k, l \in \mathbb{N}$, if $i' < l$, $j > k$, $i' + j$ is even, and $k + l$ is odd, then $\mathsf{0}^{i'}\mathsf{1}^j\mathsf{2}^k\mathsf{3}^l \in \Pi_{\mathsf{A}}$.

We must show that,

for all $j,k,l \in \mathbb{N}$, if $i < l$, $j > k$, $i+j$ is even, and $k+l$ is odd, then $\mathsf{0}^i\mathsf{1}^j\mathsf{2}^k\mathsf{3}^l \in \Pi_{\mathsf{A}}$.

Suppose $j,k,l \in \mathbb{N}$, $i < l$, $j > k$, $i+j$ is even, and $k+l$ is odd. We must show that $\mathsf{0}^i\mathsf{1}^j\mathsf{2}^k\mathsf{3}^l \in \Pi_{\mathsf{A}}$. There are three cases to consider.

($i = 0$) Since $i < l$, we have $l \geq 1$. Because $i+j$ is even, we have j is even. There are two subcases to consider.

(l **is odd**) Since $k+l$ is odd, it follows that k is even. Because j and k are even, and $j > k$, we have $j-1 > k$, $j-1 \in \mathbb{N}$ and $j-1$ is odd. Since $j-1, k \in \mathbb{N}$, $j-1 > k$, $j-1$ is odd, and k is even, $\mathsf{1}^{j-1}\mathsf{2}^k \in Y \subseteq \Pi_B$, by Lemma PS7.1.3. Since l is odd, we have $\mathsf{3}^l \in Z \subseteq \Pi_{\mathsf{C}}$, by Lemma PS7.1.2. Thus $\mathsf{0}^i\mathsf{1}^j\mathsf{2}^k\mathsf{3}^l = \mathsf{1}(\mathsf{1}^{j-1}\mathsf{2}^k)\mathsf{3}^l \in \Pi_{\mathsf{A}}$, because of production $\mathsf{A} \rightarrow \mathsf{1BC}$ of G.

(l **is even**) Because $l \geq 1$, it follows that $l \geq 2$. Thus $l-1 \in \mathbb{N}$ and $l-1$ is odd. Since $k+l$ is odd, it follows that k is odd, so that $k-1 \in \mathbb{N}$ and $k-1$ is even. Because $j > k$, we have $j-1 > k-1$, so that $j-1 \in \mathbb{N}$. Since j is even, it follows that $j-1$ is odd. Since $j-1, k-1 \in \mathbb{N}$, $j-1 > k-1$, $j-1$ is odd, and $k-1$ is even, we have $\mathsf{1}^{j-1}\mathsf{2}^{k-1} \in Y \subseteq \Pi_{\mathsf{B}}$, by Lemma PS7.1.3. Since $l-1 \in \mathbb{N}$ and $l-1$ is odd, we have $\mathsf{3}^{l-1} \in Z \subseteq \Pi_{\mathsf{C}}$, by Lemma PS7.1.2. Thus $\mathsf{0}^i\mathsf{1}^j\mathsf{2}^k\mathsf{3}^l = \mathsf{1}(\mathsf{1}^{j-1}\mathsf{2}^{k-1})\mathsf{233}^{l-1} \in \Pi_{\mathsf{A}}$, because of production $\mathsf{A} \rightarrow \mathsf{1B23C}$ of G.

($i = 1$) Since $i < l$, we have $l \geq 2$. Because $i+j$ is even, we have j is odd. There are two subcases to consider.

(l **is odd**) Since $k+l$ is odd, it follows that k is even. We have that $l-2 \in \mathbb{N}$ and $l-2$ is odd. Since $j > k$, j is odd, and k is even, we have that $\mathsf{1}^j\mathsf{2}^k \in Y \subseteq \Pi_{\mathsf{B}}$, by Lemma PS7.1.3. Since $l-2 \in \mathbb{N}$ and $l-2$ is odd, we have $\mathsf{3}^{l-2} \in Z \subseteq \Pi_{\mathsf{C}}$, by Lemma PS7.1.2. Thus $\mathsf{0}^i\mathsf{1}^j\mathsf{2}^k\mathsf{3}^l = \mathsf{0}(\mathsf{1}^j\mathsf{2}^k)\mathsf{333}^{l-2} \in \Pi_{\mathsf{A}}$, because of production $\mathsf{A} \rightarrow \mathsf{0B33C}$ of G.

(l **is even**) Since $l \geq 2$, it follows that $l-1 \in \mathbb{N}$ and $l-1$ is odd. Since $k+l$ is odd, it follows that k is odd, so that $k-1 \in \mathbb{N}$ and $k-1$ is even. Because j and k are odd, and $j > k$, we have that $j-2 > k-1$. Thus $j-2 \in \mathbb{N}$ and $j-2$ is odd. Since $j-2, k-1 \in \mathbb{N}$, $j-2 > k-1$, $j-2$ is odd, and $k-1$ is even, we have $\mathsf{1}^{j-2}\mathsf{2}^{k-1} \in Y \subseteq \Pi_{\mathsf{B}}$, by Lemma PS7.1.3. Since $l-1 \in \mathbb{N}$ and $l-1$ is odd, we have $\mathsf{3}^{l-1} \in Z \subseteq \Pi_{\mathsf{C}}$, by Lemma PS7.1.2. Thus $\mathsf{0}^i\mathsf{1}^j\mathsf{2}^k\mathsf{3}^l = \mathsf{011}(\mathsf{1}^{j-2}\mathsf{2}^{k-1})\mathsf{233}^{l-1} \in \Pi_{\mathsf{A}}$, because of production $\mathsf{A} \rightarrow \mathsf{011B23C}$ of G.

($i \geq 2$) Thus $i-2 \in \mathbb{N}$. Since $i < l$, it follows that $l \geq 2$, $i-2 < l-2$ and $l-2 \in \mathbb{N}$. Because $i+j$ is even, we have that $(i-2)+j = (i+j)-2$ is even. Since $k+l$ is odd, we have that $k+(l-2) = (k+l)-2$ is odd. Since $i-2 \in \mathbb{N}$, $i-2 < i$, $j,k,l-2 \in \mathbb{N}$, $i-2 < l-2$, $j > k$, $(i-2)+j$ is even, and $k+(l-2)$ is odd, the inductive hypothesis tells us that $\mathsf{0}^{i-2}\mathsf{1}^j\mathsf{2}^k\mathsf{3}^{l-2} \in \Pi_{\mathsf{A}}$. Thus $\mathsf{0}^i\mathsf{1}^j\mathsf{2}^k\mathsf{3}^l = \mathsf{00}(\mathsf{0}^{i-2}\mathsf{1}^j\mathsf{2}^k\mathsf{3}^{l-2})\mathsf{33} \in \Pi_{\mathsf{A}}$, because of the production $\mathsf{A} \rightarrow \mathsf{00A33}$.

□

By Lemma PS7.1.1(A) we have that $L(G) = \Pi_{\mathsf{A}} \subseteq X$. And, by Lemma PS7.1.4, we have that $X \subseteq \Pi_{\mathsf{A}} = L(G)$. Thus $L(G) = X$.

Problem 2

First, we put the text

```
{variables} A, B, C, D {start variable} A
{productions}
A -> B | C | 0A3;
B -> D | 0B2;
C -> D | 1C3;
D -> % | 1D2
```

for a grammar generating $\{\, 0^i1^j2^k3^l \mid i,j,k,l \in \mathbb{N} \text{ and } i+j = k+l \,\}$ in the file `ps7-p2-orig-gram.txt`. Next we put the text

```
{states} A, B {start state} A {accepting states} A
{transitions}
A, 0 -> B; A, 1 -> A; A, 2 -> A; A, 3 -> A;
B, 0 -> A; B, 1 -> B; B, 2 -> B; B, 3 -> B
```

for a DFA accepting all elements of $\{0, 1, 2, 3\}^*$ with an even number of 0's in the file `ps7-p2-even0s-dfa.txt`. Then we load the grammar and DFA into Forlan:

```
- val origGram = Gram.input "ps7-p2-orig-gram.txt";
val origGram = - : gram
- val even0sDFA = DFA.input "ps7-p2-even0s-dfa.txt";
val even0sDFA = - : dfa
```

Next, we put the Forlan program

```
(* standard definitions *)

val regToDFA  = nfaToDFA o efaToNFA o faToEFA o regToFA;
val minAndRen = DFA.renameStatesCanonically o DFA.minimize;

(* the alphabet {0, 1, 2, 3} *)

val syms0123 = SymSet.fromString "0, 1, 2, 3";

(* regular expression and DFA generating/accepting {0, 1, 2, 3}* *)

val allStrReg = Reg.closure(Reg.allSym syms0123);
val allStrDFA = minAndRen(regToDFA allStrReg);

(* symbolic relation on {0, 1, 2, 3} swapping 0 and 1 *)

val swap01 = SymRel.fromString "(0, 1), (1, 0), (2, 2), (3, 3)";

(* symbolic relation on {0, 1, 2, 3} swapping 0 and 2 *)

val swap02 = SymRel.fromString "(0, 2), (2, 0), (1, 1), (3, 3)";
```

```
(* symbolic relation on {0, 1, 2, 3} swapping 0 and 3 *)

val swap03 = SymRel.fromString "(0, 3), (3, 0), (1, 1), (2, 2)";

(* DFA accepting all elements of {0, 1, 2, 3}* with odd number of 1s *)

val odd1sDFA =
      minAndRen
      (DFA.minus
       (allStrDFA,
        DFA.renameAlphabet(even0sDFA, swap01)));

(* DFA accepting all elements of {0, 1, 2, 3}* with even number of 2s *)

val even2sDFA = DFA.renameAlphabet(even0sDFA, swap02);

(* DFA accepting all elements of {0, 1, 2, 3}* with odd number of 3s *)

val odd3sDFA =
      minAndRen
      (DFA.minus
       (allStrDFA,
        DFA.renameAlphabet(even0sDFA, swap03)));

(* DFA accepting all elements of {0, 1, 2, 3}* in which the
   number of 0s is even and the number of 1s is odd and the
   number of 2s is even and the number of 3s is odd *)

val paritiesDFA =
      minAndRen
      (DFA.genInter
       [even0sDFA, odd1sDFA, even2sDFA, odd3sDFA]);

(* grammar generating X *)

val gram0 =
      Gram.renameVariablesCanonically
      (Gram.inter(origGram, injDFAToEFA paritiesDFA));
```

in the file `ps7-p2-find.sml`, and proceed as follows:

```
- use "ps7-p2-find.sml";
[opening ps7-p2-find.sml]
val regToDFA = fn : reg -> dfa
val minAndRen = fn : dfa -> dfa
val syms0123 = - : sym set
val allStrReg = - : reg
val allStrDFA = - : dfa
```

```
val swap01 = - : sym_rel
val swap02 = - : sym_rel
val swap03 = - : sym_rel
val odd1sDFA = - : dfa
val even2sDFA = - : dfa
val odd3sDFA = - : dfa
val paritiesDFA = - : dfa
val gram0 = - : gram
val it = () : unit
- Gram.output("", gram0);
{variables} A, B, C, D, E, F, G, H, I {start variable} A
{productions}
A -> B; B -> F | 0C3; C -> E | 0B3; D -> H | 0E2; E -> 0D2; F -> 1G3;
G -> I | 1F3; H -> 1I2; I -> % | 1H2
val it = () : unit
```

In the grammar gram0, there are opportunities for hand-simplification using Forlan. We put the text

```
(* sumProdRHSLens : gram -> int

   sum the lengths of the right-hand sides of a grammar's
   productions *)

fun sumProdRHSLens gram =
      let fun sum nil             = 0
            | sum ((_, bs) :: ps) = length bs + sum ps
      in sum(Set.toList(Gram.productions gram)) end

(* better : gram * gram -> bool

   metric for gram1 being "better" than gram2 *)

fun better(gram1, gram2) =
      let val nv1 = Gram.numVariables gram1
          val nv2 = Gram.numVariables gram2
      in nv1 < nv2 orelse
         (nv1 = nv2 andalso
          let val np1 = Gram.numProductions gram1
              val np2 = Gram.numProductions gram2
          in np1 < np2 orelse
             (np1 = np2 andalso
              let val n1 = sumProdRHSLens gram1
                  val n2 = sumProdRHSLens gram2
              in n1 < n2 end)
          end)
      end;

(* best : gram * gram option list -> gram
```

```
   best(gram, gramOpts) returns gram if none of the optional
   grammars in gramOpts are better than gram; otherwise it returns
   one of the optional grammars that is better than gram and
   such that no other optional grammar is even better *)

fun best(gram, nil)                    = gram
  | best(gram, NONE :: gramOpts)       = best(gram, gramOpts)
  | best(gram, SOME gram' :: gramOpts) =
      if better(gram', gram)
      then best(gram', gramOpts)
      else best(gram, gramOpts);

(* elims : gram -> gram

   recursively eliminate variables of a grammar *)

fun elims gram =
      let val qs       =
            SymSet.minus
            (Gram.variables gram, Set.sing(Gram.startVariable gram))
          val gramOpts =
            map (fn q =>
                   case Gram.eliminateVariableOpt(gram, q) of
                        NONE       => NONE
                      | SOME gram' => SOME(elims gram'))
                (Set.toList qs)
      in best(gram, gramOpts) end;

(* handSimp : gram -> gram

   hand-simplify a grammar *)

fun handSimp gram =
      let val gram = elims gram
      in case Gram.restartOpt gram of
              NONE       => gram
            | SOME gram' => gram'
      end;
```

in the file `ps7-p2-hand-simp.sml`, and proceed as follows to obtain our answer, `gram`:

```
- use "ps7-p2-hand-simp.sml";
[opening ps7-p2-hand-simp.sml]
val sumProdRHSLens = fn : gram -> int
val better = fn : gram * gram -> bool
val best = fn : gram * gram option list -> gram
val elims = fn : gram -> gram
val handSimp = fn : gram -> gram
```

```
val it = () : unit
- val gram = Gram.renameVariablesCanonically(handSimp gram0);
val gram = - : gram
- Gram.output("", gram);
{variables} A, B, C, D {start variable} A
{productions}
A -> 0B3 | 1C3 | 00A33; B -> 00B22 | 01D22; C -> D | 11C33; D -> % | 11D22
val it = () : unit
- (Gram.numVariables gram, Gram.numProductions gram);
val it = (4,9) : int * int
```

Problem 3

Suppose, toward a contradiction, that X is context-free. Thus there is an $n \in \mathbb{N} - \{0\}$ with the property of the Pumping Lemma for Context-free Languages, where X has been substituted for L. Let $z = \mathsf{0}^n \mathsf{1}^{n+1} \mathsf{2}^{n+2}$. Since $n < n+1 < n+2$, we have that $z \in X$. Furthermore $|z| = 3n+3 \geq n$, and thus the property of the lemma tells us there are $u, v, w, x, y \in \mathbf{Str}$ such that $z = uvwxy$ and

(1) $|vwx| \leq n$; and

(2) $vx \neq \%$; and

(3) $uv^iwx^iy \in X$, for all $i \in \mathbb{N}$.

Because $\mathsf{0}^n \mathsf{1}^{n+1} \mathsf{2}^{n+2} = z = uvwxy$, we have that $u, v, w, x, y \in \{\mathsf{0}, \mathsf{1}, \mathsf{2}\}^*$, and (1) tells us that vwx does not have both 0's and 2's. There are four cases to consider:

(vx has one or more 0's, and has one or more 1's) Then vx has no 2's. Let k be the number of 1's in vx, so that $k \geq 1$. By (3), we have that $uvvwxxy = uv^2wx^2y \in X$. Since $uvwxy$ has $n+1$ 1's and $n+2$ 2's, $u(v)vwx(x)y$ has $n+1+k$ 1's, and has $n+2$ 2's. But $n+1+k \geq n+2$, so that $uvvwxxy \notin X$—contradiction.

(vx has one or more 0's, but has no 1's) Let k be the number of 0's in vx, so that $k \geq 1$. By (3), we have that $uvvwxxy = uv^2wx^2y \in X$. Since $uvwxy$ has n 0's and $n+1$ 1's, $u(v)vwx(x)y$ has $n+k$ 0's, and has $n+1$ 1's. But $n+k \geq n+1$, so that $uvvwxxy \notin X$—contradiction.

(vx has no 0's, but has one or more 1's) Let k be the number of 1's in vx, so that $k \geq 1$. By (3), we have that $uwy = uv^0wx^0y \in X$. Since $u(v)w(x)y$ has n 0's and $n+1$ 1's, uwy has n 0's, and has $n+1-k$ 1's. But $n \geq n+1-k$, so that $uwy \notin X$—contradiction.

(vx has no 0's, and has no 1's) By (2), it follows that vx has one or more 2's. Let k be the number of 2's in vx, so that $k \geq 1$. By (3), we have that $uwy = uv^0wx^0y \in X$. Since $u(v)w(x)y$ has $n+1$ 1's and $n+2$ 2's, uwy has $n+1$ 1's, and has $n+2-k$ 2's. But $n+1 \geq n+2-k$, so that $uwy \notin X$—contradiction.

Because we obtained a contradiction in each case, we have an overall contradiction. Thus X is not context-free.