CS 591 S2—Formal Language Theory: Integrating Experimentation and Proof—Fall 2019

Final Examination

Model Answers

Question 1

We can conclude that:

- if $j = 0$, then $j' = 0$ and $i + k = i' + k'$; and

- if $j \geq 1$, then $i = i'$, $j = j'$ and $k = k'$.

Question 2
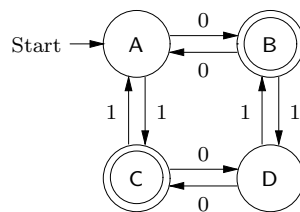
$$A \to A2 \mid B2 \mid D2,$$
$$B \to 0B2 \mid 0C2,$$
$$C \to 1C2 \mid \%,$$
$$D \to 0D2 \mid E,$$
$$E \to 1E2 \mid 12.$$

Question 3

M is

First, we show by induction on Λ that:

(A) for all $w \in \Lambda_{\mathsf{A}}$, $w \in X_{\mathsf{ee}}$;

(B) for all $w \in \Lambda_{\mathsf{B}}$, $w \in X_{\mathsf{oe}}$;

(C) for all $w \in \Lambda_{\mathsf{C}}$, $w \in X_{\mathsf{eo}}$;

(D) for all $w \in \Lambda_\mathsf{D}$, $w \in X_\mathsf{oo}$.

There are nine (one plus the number of transitions) parts to show:

(empty string)   We must show that $\% \in X_\mathsf{ee}$, and this follows by fact (1).

(A, 0 → B)   Suppose $w \in \Lambda_\mathsf{A}$, and assume the inductive hypothesis, $w \in X_\mathsf{ee}$. Then $w0 \in X_\mathsf{ee}\{0\} \subseteq X_\mathsf{oe}$, by fact (2).

(A, 1 → C)   Suppose $w \in \Lambda_\mathsf{A}$, and assume the inductive hypothesis, $w \in X_\mathsf{ee}$. Then $w1 \in X_\mathsf{ee}\{1\} \subseteq X_\mathsf{eo}$, by fact (3).

(B, 0 → A)   Suppose $w \in \Lambda_\mathsf{B}$, and assume the inductive hypothesis, $w \in X_\mathsf{oe}$. Then $w0 \in X_\mathsf{oe}\{0\} \subseteq X_\mathsf{ee}$, by fact (6).

(B, 1 → D)   Suppose $w \in \Lambda_\mathsf{B}$, and assume the inductive hypothesis, $w \in X_\mathsf{oe}$. Then $w1 \in X_\mathsf{oe}\{1\} \subseteq X_\mathsf{oo}$, by fact (7).

(C, 0 → D)   Suppose $w \in \Lambda_\mathsf{C}$, and assume the inductive hypothesis, $w \in X_\mathsf{eo}$. Then $w0 \in X_\mathsf{eo}\{0\} \subseteq X_\mathsf{oo}$, by fact (4).

(C, 1 → A)   Suppose $w \in \Lambda_\mathsf{C}$, and assume the inductive hypothesis, $w \in X_\mathsf{eo}$. Then $w1 \in X_\mathsf{eo}\{1\} \subseteq X_\mathsf{ee}$, by fact (5).

(D, 0 → C)   Suppose $w \in \Lambda_\mathsf{D}$, and assume the inductive hypothesis, $w \in X_\mathsf{oo}$. Then $w0 \in X_\mathsf{oo}\{0\} \subseteq X_\mathsf{eo}$, by fact (8).

(D, 1 → B)   Suppose $w \in \Lambda_\mathsf{D}$, and assume the inductive hypothesis, $w \in X_\mathsf{oo}$. Then $w1 \in X_\mathsf{oo}\{1\} \subseteq X_\mathsf{oe}$, by fact (9).

Now we use the result of our induction on Λ to show that $L(M) = X_\mathsf{eo} \cup X_\mathsf{oe}$.

$(L(M) \subseteq X_\mathsf{eo} \cup X_\mathsf{oe})$ Suppose $w \in L(M)$. Because $A_M = \{\mathsf{B}, \mathsf{C}\}$, we have that $w \in L(M) = \Lambda_\mathsf{B} \cup \Lambda_\mathsf{C}$. Thus there are two cases to consider:

- Suppose $w \in \Lambda_\mathsf{B}$. By part (B) of our induction on Λ, we have $w \in X_\mathsf{oe} \subseteq X_\mathsf{eo} \cup X_\mathsf{oe}$.

- Suppose $w \in \Lambda_\mathsf{C}$. By part (C) of our induction on Λ, we have $w \in X_\mathsf{eo} \subseteq X_\mathsf{eo} \cup X_\mathsf{oe}$.

$(X_\mathsf{eo} \cup X_\mathsf{oe} \subseteq L(M))$ Suppose $w \in X_\mathsf{eo} \cup X_\mathsf{oe}$. Since $X_\mathsf{eo} \cup X_\mathsf{oe} \subseteq \{0, 1\}^*$, we have that $w \in \{0, 1\}^*$. Suppose, toward a contradiction, that $w \notin L(M)$. Because $w \notin L(M) = \Lambda_\mathsf{B} \cup \Lambda_\mathsf{C}$, and $w \in \{0, 1\}^* = (\textbf{alphabet}\, M)^* = \Lambda_\mathsf{A} \cup \Lambda_\mathsf{B} \cup \Lambda_\mathsf{C} \cup \Lambda_\mathsf{D}$, it follows that $w \in \Lambda_\mathsf{A} \cup \Lambda_\mathsf{D}$. Thus there are two cases to consider:

- Suppose $w \in \Lambda_\mathsf{A}$. By part (A) of our induction on Λ, we have $w \in X_\mathsf{ee}$. Thus **zeros** w and **ones** w are both even. Since **ones** w is even, we have $w \notin X_\mathsf{eo}$. Since **zeros** w is even, we have $w \notin X_\mathsf{oe}$. Thus $w \notin X_\mathsf{eo} \cup X_\mathsf{oe}$—contradiction.

- Suppose $w \in \Lambda_{\mathsf{D}}$. By part (D) of our induction on Λ, we have $w \in X_{\mathsf{oo}}$. Thus $\mathbf{zeros}\, w$ and $\mathbf{ones}\, w$ are both odd. Since $\mathbf{zeros}\, w$ is odd, we have $w \notin X_{\mathsf{eo}}$. Since $\mathbf{ones}\, w$ is odd, we have $w \notin X_{\mathsf{oe}}$. Thus $w \notin X_{\mathsf{eo}} \cup X_{\mathsf{oe}}$—contradiction.

Because we obtained a contradiction in both cases, we have an overall contradiction. Thus $w \in L(M)$.

Question 4

Suppose, toward a contradiction, that X is regular. Thus there is an $n \in \mathbb{N} - \{0\}$ with the property of the Pumping Lemma for Regular Languages, where X has been substituted for L. Let $z = 0^{2n}1^{1}2^{2n}3^{1}$. Because $2n + 1 = 2n + 1$, $2n$ is even, 1 is odd, $2n$ is even, and 1 is odd, we have that $z \in X$. And $|z| = 4n + 2 \geq n$. Thus the property of the pumping lemma tells us that there are $u, v, w \in \mathbf{Str}$ such that $z = uvw$ and

(1) $|uv| \leq n$; and

(2) $v \neq \%$; and

(3) $uv^{i}w \in X$, for all $i \in \mathbb{N}$.

Since $0^{2n}1^{1}2^{2n}3^{1} = z = uvw$, (1) tells us that uv consists of only 0's. Thus (2) tells us that $v = 0^{p}$ for some $p \geq 1$. Consequently, uw has: $2n - p$ occurrences of 0; 1 occurrence of 1; $2n$ occurrences of 2; and 1 occurrence of 3. Thus the sum of the numbers of occurrences of 0 and 1 in uw is $(2n - p) + 1 = (2n + 1) - p$, whereas the sum of the numbers of occurrences of 2 and 3 in uw is $2n + 1$. But according to (3), $uw = uv^{0}w \in X$, with the consequence that $(2n + 1) - p = 2n + 1$. But then $p = 0$—contradiction. Thus X is not regular.

Question 5

First, we convert the regular expression $(0 + 1)^{*}$ generating $\{0, 1\}^{*}$ into a minimized DFA **allStrDFA**. (We first convert it to an FA, then to an EFA, then to an NFA, and then to a DFA, at which point we can minimize the DFA.) Then $L(\mathbf{allStrDFA}) = L((0 + 1)^{*}) = \{0, 1\}^{*}$. Next, we convert α into a minimized DFA $\mathbf{isDFA}_{\alpha}$, and convert β into a minimized DFA $\mathbf{isDFA}_{\beta}$. Then we let the EFA **compEFA** be $\mathbf{concat}(\mathbf{isDFA}_{\alpha}, \mathbf{isDFA}_{\beta})$. Let

$$Y = \{\, w \in \{0, 1\}^{*} \mid \text{there are } x, y \in \{0, 1\}^{*} \text{ such that } w = xy \text{ and } x \in L(\alpha) \text{ and } y \in L(\beta) \,\}.$$

It is easy to check that $L(\mathbf{compEFA}) = L(\alpha)\, L(\beta) = Y$. Let **compDFA** be the result of converting **compEFA** to a minimized DFA. Next, let $\mathbf{ansDFA} = \mathbf{minus}(\mathbf{allStrDFA}, \mathbf{compDFA})$. Then $L(\mathbf{ansDFA}) = \{0, 1\}^{*} - Y = X$. Finally, we convert **ansDFA** into a regular expression, using our weak simplification function on regular expressions, producing γ. Hence $L(\gamma) = X$.