Chapter 2: Formal Languages

In this chapter, we

- say what symbols, strings, alphabets and (formal) languages are,
- show how to use various induction principles to prove language equalities, and
- give an introduction to the Forlan toolset.

In subsequent chapters, we will study four more restricted kinds of languages: the regular (Chapter 3), context-free (Chapter 4), recursive and recursively enumerable (Chapter 5) languages.

2.1: Symbols, Strings, Alphabets and (Formal) Languages

In this section, we define the basic notions of the subject: symbols, strings, alphabets and (formal) languages.

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Raising a String to a Power

We define the string x^n resulting from *raising* a string x *to a power* $n \in \mathbb{N}$ by recursion on n:

$$x^0 = \%, \text{ for all } x \in \textbf{Str};$$
$$x^{n+1} = xx^n, \text{ for all } x \in \textbf{Str} \text{ and } n \in \mathbb{N}.$$

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Proposition 2.1.2
For all $x \in \textbf{Str}$ and $n, m \in \mathbb{N}$, $x^{n+m} = x^n x^m$.

Proof. An easy mathematical induction on n. The string x and the natural number m can be fixed at the beginning of the proof.
□

Prefixes, Suffixes and Substrings

Suppose x and y are strings. We say that:

- x is a *prefix* of y iff $y = xv$ for some $v \in \textbf{Str}$;
- x is a *suffix* of y iff $y = ux$ for some $u \in \textbf{Str}$;
- x is a *substring* of y iff $y = uxv$ for some $u, v \in \textbf{Str}$.

A prefix, suffix or substring of a string other than the string itself is called *proper*.

For example:

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If Σ is an alphabet, then we write Σ^* for **List** Σ.

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Here are some example languages (all are $\{0,1\}$-languages):

- $\emptyset$;
- $\{0,1\}^*$;
- $\{010, 1001, 1101\}$;
- $\{\, 0^n1^n \mid n \in \mathbb{N} \,\}$;
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For example, $\textbf{Sym}^* = \textbf{Str}$.