

## 3.6: *Checking Acceptance and Finding Accepting Paths*

In this section we study algorithms for:

- checking whether a string is accepted by a finite automaton;  
and
- finding a labeled path that explains why a string is accepted by a finite automaton.

## *Processing a String from a Set of States*

Suppose  $M$  is a finite automaton. We define a function  $\Delta_M \in \mathcal{P} Q_M \times \mathbf{Str} \rightarrow \mathcal{P} Q_M$  by:  $\Delta_M(P, w)$  is the set of all  $r \in Q_M$  such that there is an  $lp \in \mathbf{LP}$  such that

- $w$  is the label of  $lp$ ;
- $lp$  is valid for  $M$ ;
- the start state of  $lp$  is in  $P$ ; and
- $r$  is the end state of  $lp$ .

## *Processing a String from a Set of States*

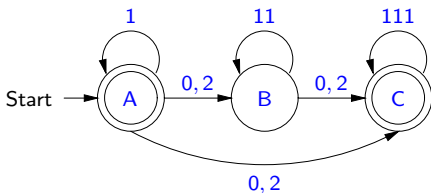
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When the FA  $M$  is clear from the context, we sometimes abbreviate  $\Delta_M$  to  $\Delta$ .

## $\Delta$ Function Examples

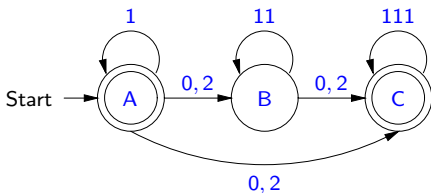
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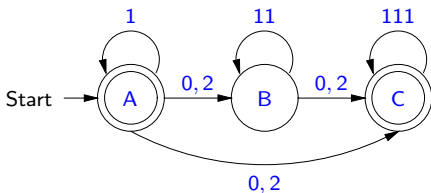
Then,  $\Delta_M(\{A\}, 12111111) = \{B, C\}$ , since

$$A \xrightarrow{1} A \xrightarrow{2} B \xrightarrow{11} B \xrightarrow{11} B \xrightarrow{11} B \quad \text{and} \quad A \xrightarrow{1} A \xrightarrow{2} C \xrightarrow{111} C \xrightarrow{111} C$$

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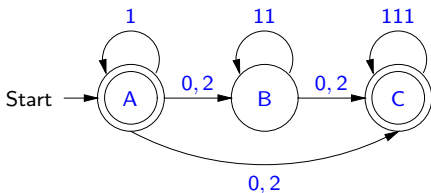
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are all of the labeled paths that are labeled by  $11$  and valid for  $M$ .

## *An Algorithm for Calculating $\Delta(P, w)$*

Suppose  $M$  is a finite automaton,  $P \subseteq Q_M$  and  $w \in \mathbf{Str}$ . We can calculate  $\Delta_M(P, w)$  as follows.

Let  $S$  be the set of all suffixes of  $w$ . Given  $y \in S$ , we write **pre**  $y$  for the unique  $x$  such that  $w = xy$ .

First, we generate the least subset  $X$  of  $Q_M \times S$  such that:



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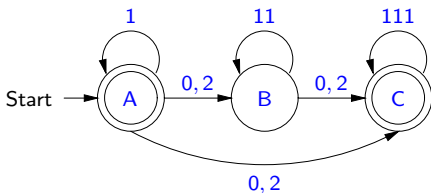
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## Calculating $\Delta(P, w)$

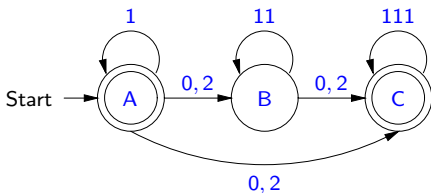
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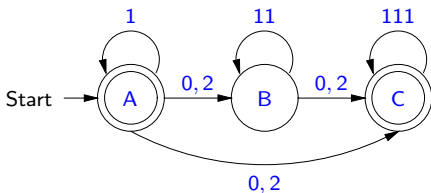


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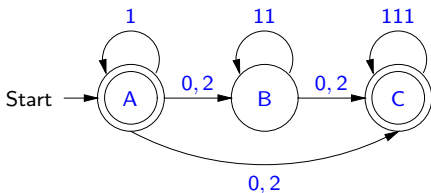


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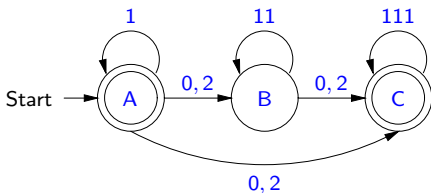


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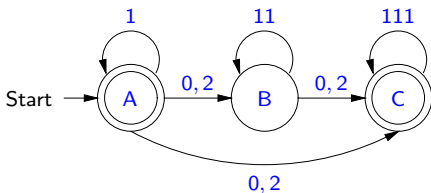
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- $(C, \%)$ , because of the transition  $C, 111 \rightarrow C$ .

## Calculating $\Delta(P, w)$

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But  $\text{pre } w = \%$ , so that  $p \in \Delta_M(P, \text{pre } w)$ .
- (2) Suppose  $q, r \in Q_M$ ,  $x, y \in \mathbf{Str}$ ,  $(q, xy) \in X$  and  $(q, x, r) \in T_M$ . Assume the inductive hypothesis:  
 $q \in \Delta_M(P, \text{pre}(xy))$ .

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## Calculating $\Delta(P, w)$

**Proof (cont.).** For the ‘if’ (right-to-left) direction, we have that there is a labeled path

$$q_1 \xRightarrow{x_1} q_2 \xRightarrow{x_2} \cdots q_n \xRightarrow{x_n} q_{n+1},$$

that is valid for  $M$  and where  $\text{pre } y = x_1x_2 \cdots x_n$ ,  $q_1 \in P$  and  $q_{n+1} = q$ . Since  $q_1 \in P$  and  $w = (\text{pre } y)y = x_1x_2 \cdots x_ny$ , we have that  $(q_1, x_1x_2 \cdots x_ny) = (q_1, w) \in X$ , by (1). But  $(q_1, x_1, q_2) \in T_M$ , and thus  $(q_2, x_2 \cdots x_ny) \in X$ , by (2). Continuing on in this way (we could do this by mathematical induction), we finally get that  $(q, y) = (q_{n+1}, y) \in X$ .  $\square$

## Calculating $\Delta(P, w)$

### Lemma 3.6.2

For all  $q \in Q_M$ ,  $(q, \% ) \in X$  iff  $q \in \Delta_M(P, w)$ .

**Proof.** Suppose  $(q, \% ) \in X$ . Lemma 3.6.1 tells us that  $q \in \Delta_M(P, \mathbf{pre} \% )$ . But  $\mathbf{pre} \% = w$ , and thus  $q \in \Delta_M(P, w)$ .

Suppose  $q \in \Delta_M(P, w)$ . Since  $w = \mathbf{pre} \%$ , we have that  $q \in \Delta_M(P, \mathbf{pre} \% )$ . Lemma 3.6.1 tells us that  $(q, \% ) \in X$ .  $\square$

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By Lemma 3.6.2, we have that

$$\Delta_M(P, w) = \{ q \in Q_M \mid (q, \% ) \in X \}.$$

Thus, we return the set of all states  $q$  that are paired with  $\%$  in  $X$ .

# Checking String Acceptance and Finding Accepting Paths

## Proposition 3.6.3

Suppose  $M$  is a finite automaton. Then

$$L(M) = \{ w \in \mathbf{Str} \mid \Delta_M(\{s_M\}, w) \cap A_M \neq \emptyset \}.$$

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Given a finite automaton  $M$ , subsets  $P, R$  of  $Q_M$  and a string  $w$ , how do we search for a labeled path that is labeled by  $w$ , valid for  $M$ , starts from an element of  $P$ , and ends with an element of  $R$ ?  
What we need to do is associate with each pair

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With a bit of care, we can ensure that these labeled paths are as short as possible.

As we generate the elements of  $X$ , we look for a pair of the form

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As we generate the elements of  $X$ , we look for a pair of the form  $(q, y)$ , where  $q \in R$ . Our answer will then be the labeled path associated with this pair.

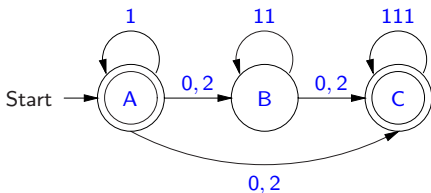
## *Checking Acceptance in Forlan*

The Forlan module **FA** also contains the following functions for processing strings, checking string acceptance, and finding labeled paths:

```
val processStr      : fa -> sym set * str -> sym set
val accepted       : fa -> str -> bool
val findLP         : fa -> sym set * str * sym set -> lp
val findAcceptingLP : fa -> str -> lp
```

## Forlan Examples

Suppose `fa` is the finite automaton



We begin by applying our four functions to `fa`, and giving names to the resulting functions:

```
- val processStr = FA.processStr fa;
  val processStr = fn : sym set * str -> sym set
- val accepted = FA.accepted fa;
  val accepted = fn : str -> bool
```

## *Forlan Examples*

Continuing:

```
- val findLP = FA.findLP fa;  
val findLP = fn : sym set * str * sym set -> lp  
- val findAcceptingLP = FA.findAcceptingLP fa;  
val findAcceptingLP = fn : str -> lp
```

Next, we'll define a set of states and a string to use later:

```
- val bs = SymSet.input "";  
@ A, B, C  
@ .  
val bs = - : sym set  
- val x = Str.input "";  
@ 11  
@ .  
val x = [-,-] : str
```



## *Forlan Examples*

Here are some example uses of our functions:

```
- SymSet.output("", processStr(bs, x));  
A, B  
val it = () : unit  
- accepted(Str.input "");  
@ 12111111  
@ .  
val it = true : bool  
- accepted(Str.input "");  
@ 1211  
@ .  
val it = false : bool
```

## *Forlan Examples*

More examples:

```
- LP.output("", findLP(bs, x, bs));  
B, 11 => B  
val it = () : unit  
- LP.output("", findAcceptingLP(Str.input ""));  
@ 12111111  
@ .  
A, 1 => A, 2 => C, 111 => C, 111 => C  
val it = () : unit  
- LP.output("", findAcceptingLP(Str.input ""));  
@ 222  
@ .  
no such labeled path exists  
  
uncaught exception Error
```