

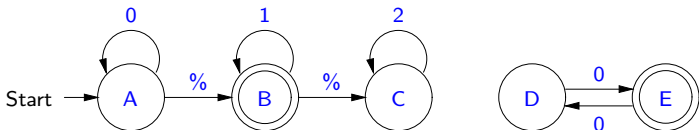
3.7: Simplification of Finite Automata

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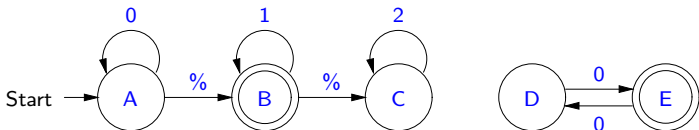


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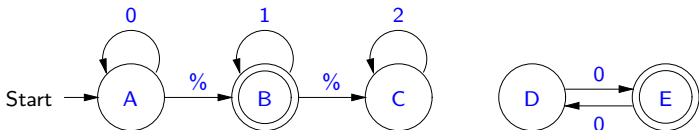
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What is odd about M ?

First, there are no valid labeled paths from the start state to D and E , and so these states are redundant.

Second, there are no valid labeled paths from C to an accepting state, and so it is also redundant.

Useful States

Suppose M is a finite automaton. We say that a state $q \in Q_M$ is:

- *reachable in M* iff there is a labeled path lp such that lp is valid for M , the start state of lp is s_M , and the end state of lp is q ;

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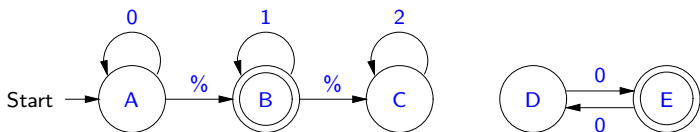
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- *dead in M* iff q is not live in M ; and
- *useful in M* iff q is both reachable and live in M .

Useful States Example

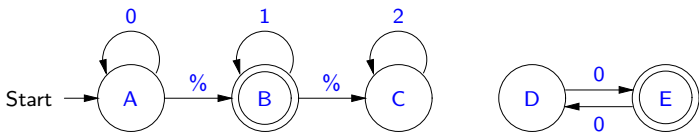
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The reachable states of M are:

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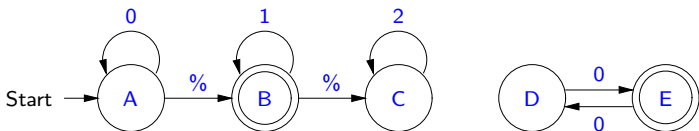
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The reachable states of M are: A , B and C . The live states of M are:

Useful States Example

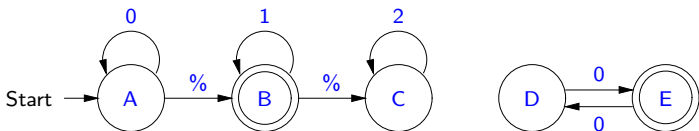
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Useful States Example

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The reachable states of M are: A , B and C . The live states of M are: A , B , D and E . And, the useful states of M are: A and B .

Generating Reachable, Live and Useful States

There is a simple algorithm for generating the set of reachable states of a finite automaton M . We generate the least subset X of Q_M such that:

- $q_0 \in X$; and
- for all $q, r \in Q_M$ and $x \in \mathbf{Str}$, if $q \in X$ and $(q, x, r) \in T_M$, then $r \in X$.

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Similarly, there is a simple algorithm for generating the set of live states of a finite automaton M . We generate the least subset Y of Q_M such that:

- $\subseteq Y$; and
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Similarly, there is a simple algorithm for generating the set of live states of a finite automaton M . We generate the least subset Y of Q_M such that:

- $A_M \subseteq Y$; and
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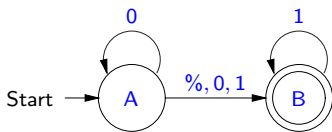
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Thus, we can generate the set of useful states of an FA by generating the set of reachable states, generating the set of live states, and intersecting those sets of states.

Redundant Transitions

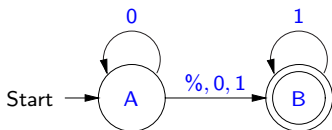
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What is odd about this machine?

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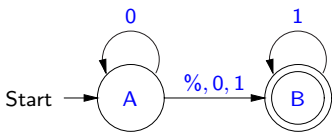


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Here, the transitions $(A, 0, B)$ and $(A, 1, B)$ are redundant.

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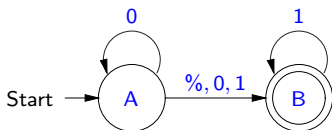
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Given an FA M and a finite subset U of $\{(q, x, r) \mid q, r \in Q_M \text{ and } x \in \mathbf{Str}\}$, we write M/U for the FA that is identical to M except that its set of transitions is U .

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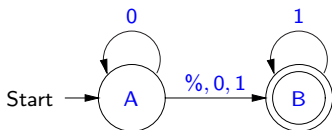
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If M is an FA and $(p, x, q) \in T_M$, we say that:

- (p, x, q) is redundant in M iff $q \in \Delta_N(\{p\}, x)$, where $N = M/(T_M - \{(p, x, q)\})$; and

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- (p, x, q) is *redundant* in M iff $q \in \Delta_N(\{p\}, x)$, where $N = M/(T_M - \{(p, x, q)\})$; and
- (p, x, q) is *irredundant* in M iff (p, x, q) is not redundant in M .

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We say that a finite automaton M is *simplified* iff either

- every state of M is useful, and every transition of M is irredundant; or

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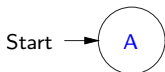
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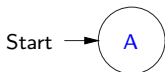


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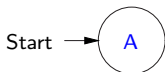
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Proposition 3.7.1

If M is a simplified finite automaton, then

alphabet($L(M)$) = **alphabet** M .

Algorithm for Removing Redundant Transitions

Given an FA M , $p, q \in Q_M$ and $x \in \mathbf{Str}$, we say that (p, x, q) is *implicit in M* iff $q \in \Delta_M(\{p\}, x)$.

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Given an FA M , we define a function $\mathbf{remRedun}_M \in \mathcal{P} T_M \times \mathcal{P} T_M \rightarrow \mathcal{P} T_M$ by well-founded recursion on the size of its second argument.

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- Otherwise, let v be the greatest element of $\{(q, x, r) \in V \mid \text{there are no } q', r' \in \mathbf{Sym} \text{ and } y \in \mathbf{Str} \text{ such that } (q', y, r') \in V \text{ and } |y| > |x|\}$, and $V' = V - \{v\}$.

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Algorithm for Removing Redundant Transitions

In general, there are multiple—incompatible—ways of removing redundant transitions from an FA. **remRedun** is defined so as to favor removing transitions whose labels are longer; and among transitions whose labels have equal length, to favor removing transitions that are larger in our total ordering on transitions.

Simplification Algorithm

We define a function **simplify** \in **FA** \rightarrow **FA** by: **simplify** M is the finite automaton N produced by the following process.

- First, the useful states are M are determined.

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 - $Q_N = \{s_M\}$;
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 - $T_N = \emptyset$.
- And, if s_M is useful in M , then N is **remRedun** $_{N'}(\emptyset, T_{N'})$, where N' is defined by
 - $Q_{N'} = \{q \in Q_M \mid q \text{ is useful in } M\}$;
 - $s_{N'} = s_M$;
 - $A_{N'} = A_M \cap Q_{N'} = \{q \in A_M \mid q \in Q_{N'}\}$; and
 - $T_{N'} = \{(q, x, r) \in T_M \mid q, r \in Q_{N'}\}$.

More on Simplification Algorithm

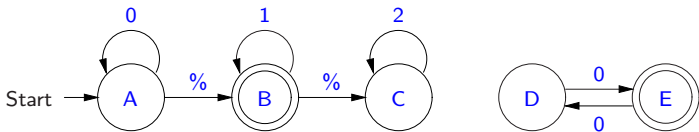
Proposition 3.7.3

Suppose M is a finite automaton. Then:

- (1) **simplify** M is simplified;
- (2) **simplify** $M \approx M$;
- (3) $Q_{\text{simplify } M} \subseteq Q_M$ and $T_{\text{simplify } M} \subseteq T_M$; and
- (4) **alphabet**(**simplify** M) = **alphabet**($L(M)$) \subseteq **alphabet** M .

Simplification Examples

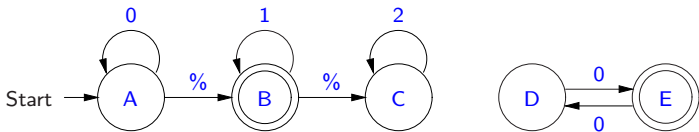
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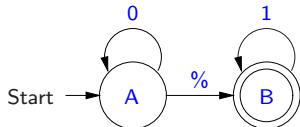
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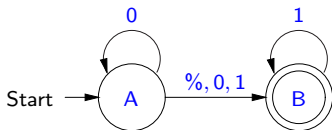


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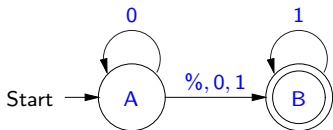
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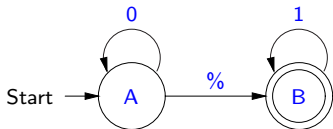
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Testing Whether $L(M) = \emptyset$

Our simplification algorithm gives us an algorithm for testing whether the language accepted by an FA M is empty. We first simplify M , calling the result N . We then test whether

Testing Whether $L(M) = \emptyset$

Our simplification algorithm gives us an algorithm for testing whether the language accepted by an FA M is empty. We first simplify M , calling the result N . We then test whether $A_N = \emptyset$. If the answer is “yes”, clearly $L(M) = L(N) = \emptyset$. And if the answer is “no”, then s_N is useful, and so N (and thus M) accepts at least one string.

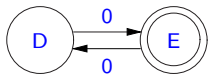
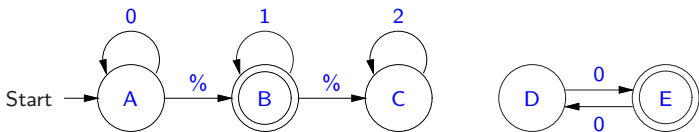
Simplification in Forlan

The Forlan module FA includes the following functions relating to the simplification of finite automata:

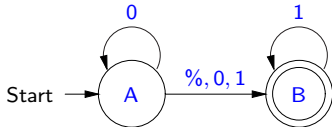
```
val simplify    : fa -> fa  
val simplified  : fa -> bool
```

Simplification Examples in Forlan

In the following, suppose **fa1** is the finite automaton

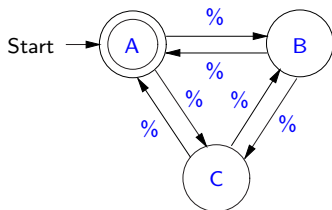


fa2 is the finite automaton



Simplification Examples in Forlan

and **fa3** is the finita automaton



Simplification Examples in Forlan

Here are some example uses of `simplify` and `simplified`:

```
- FA.simplified fa1;
val it = false : bool
- val fa1' = FA.simplify fa1;
val fa1' = - : fa
- FA.output("", fa1');
{states} A, B {start state} A {accepting states} B
{transitions} A, % -> B; A, 0 -> A; B, 1 -> B
val it = () : unit
- FA.simplified fa1';
val it = true : bool
- val fa2' = FA.simplify fa2;
val fa2' = - : fa
- FA.output("", fa2');
{states} A, B {start state} A {accepting states} B
{transitions} A, % -> B; A, 0 -> A; B, 1 -> B
val it = () : unit
```

Simplification Examples in Forlan

```
- val fa3' = FA.simplify fa3;  
val fa3' = - : fa  
- FA.output("", fa3');  
{states} A, B, C {start state} A {accepting states} A  
{transitions} A, % -> B | C; B, % -> A; C, % -> A  
val it = () : unit
```

Thus the simplification of `fa3` resulted in the removal of the %-transitions between `B` and `C`.