3.8: Proving the Correctness of Finite Automata

In this section, we consider techniques for proving the correctness of finite automata, i.e., for proving that finite automata accept the languages we want them to.

Properties of Δ

Proposition 3.8.1

Suppose M *is a finite automaton.*

(1) *For all* $q \in Q_M$, $\quad \in \Delta_M(\{q\}, \%)$.

(2) *For all* $q, r \in Q_M$ *and* $w \in \mathbf{Str}$, *if* $q, w \to r \in T_M$, *then* $\quad \in \Delta_M(\{q\}, w)$.

(3) *For all* $p, q, r \in Q_M$ *and* $x, y \in \mathbf{Str}$, *if* $q \in \Delta_M(\{p\}, x)$ *and* $r \in \Delta_M(\{q\}, y)$, *then* $\quad \in \Delta_M(\{p\}, xy)$.

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Definition of Λ

Suppose M is a finite automaton and $q \in Q_M$. Then we define

$$\Lambda_{M,q} = \{\, w \in \mathbf{Str} \mid q \in \Delta_M(\{s_M\}, w) \,\}.$$

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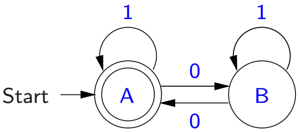
Clearly, $\Lambda_{M,q} \subseteq (\mathbf{alphabet}\, M)^*$, for all FAs M and $q \in Q_M$.

If it's clear which FA we are talking about, we sometimes abbreviate $\Lambda_{M,q}$ to Λ_q.

Λ *Example*

Let our example FA, M, be



Then:

- $01101 \in \Lambda_A$, because of the labeled path

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For all FA M,

$$L(M) = \bigcup \{\, \Lambda_{M,q} \mid q \in A_M \,\},$$

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Example Finite Automaton

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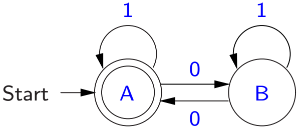

Let

$$X = \{\, w \in \{0,1\}^* \mid w \text{ has an even number of } 0\text{'s} \,\}$$
$$Y = \{\, w \in \{0,1\}^* \mid w \text{ has an odd number of } 0\text{'s} \,\}.$$

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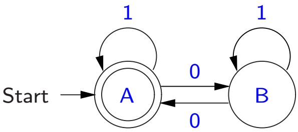

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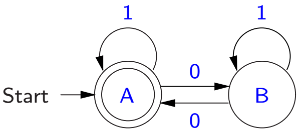

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But our approach will also involve showing $\Lambda_{M,\mathsf{B}} = Y$. We would cope with more states analogously, having one language per state.

Proving that Enough is Accepted

First we study techniques for showing that everything we want an automaton to accept is really accepted.

Since $X, Y \subseteq \{0,1\}^*$, to prove that $X \subseteq \Lambda_{M,\mathsf{A}}$ and $Y \subseteq \Lambda_{M,\mathsf{B}}$, it will suffice to use strong string induction to show that, for all $w \in \{0,1\}^*$:

(A) if $w \in X$, then $w \in \Lambda_{M,\mathsf{A}}$; and

(B) if $w \in Y$, then $w \in \Lambda_{M,\mathsf{B}}$.

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We proceed by strong string induction. Suppose $w \in \{0,1\}^*$, and assume the inductive hypothesis: for all $x \in \{0,1\}^*$, if x is a proper substring of w, then:

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- Suppose $w = x1$, for some $x \in \{0, 1\}^*$. Thus x has an even number of 0's, so that $x \in X$. Because x is a proper substring of w, Part (A) of the inductive hypothesis tells us that $x \in \Lambda_{\mathsf{A}}$. Furthermore, $\mathsf{A}, 1 \rightarrow \mathsf{A} \in T$, so that $w = x1 \in \Lambda_{\mathsf{A}}$, by Proposition 3.8.3(2).

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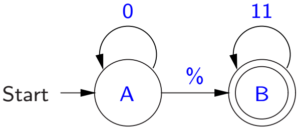
(B) This case is symmetric to (A), and is in the book.

%-*Transitions and Enough is Accepted*

Let N be the finite automaton



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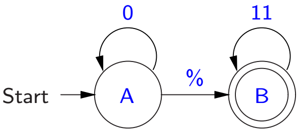

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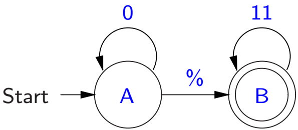

Here we hope that $\Lambda_{N,\mathsf{A}} = \{0\}^*$ and $L(N) = \Lambda_{N,\mathsf{B}} = \{0\}^*\{11\}^*$

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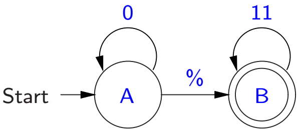

Here we hope that $\Lambda_{N,\mathsf{A}} = \{0\}^*$ and $L(N) = \Lambda_{N,\mathsf{B}} = \{0\}^*\{11\}^*$, but if we try to prove that

$$\begin{aligned} \{0\}^* &\subseteq \Lambda_{N,\mathsf{A}}, \\ \{0\}^*\{11\}^* &\subseteq \Lambda_{N,\mathsf{B}} \end{aligned}$$

using our standard technique, there is a complication related to the %-transition.

We use strong string induction to show that, for all $w \in \{0,1\}^*$:

(A) if $w \in \{0\}^*$, then $w \in \Lambda_{\mathsf{A}}$;

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In Part (B), we assume that $w \in \{0\}^*\{11\}^*$, so that $w = 0^n(11)^m$ for some $n, m \in \mathbb{N}$. We must show that $w \in \Lambda_{\mathsf{B}}$. We consider two cases: $m = 0$ and $m \geq 1$. The second of these is straightforward, so let's focus on the first. Then $w = 0^n \in \{0\}^*$. We want to use Part (A) of the inductive hypothesis to conclude that $0^n \in \Lambda_{\mathsf{A}}$, but there is a problem:

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So, we must consider two subcases, when $n = 0$ and $n \geq 1$. In the first subcase, because $\% \in \Lambda_{\mathsf{A}}$ and $\mathsf{A}, \% \to \mathsf{B} \in T$, we have that $w = \% = \%\% \in \Lambda_{\mathsf{B}}$.

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In the second subcase, we have that $w = 0^{n-1}0$. By Part (A) of the inductive hypothesis, we have that $0^{n-1} \in \Lambda_\mathsf{A}$. Thus, because $\mathsf{A}, 0 \rightarrow \mathsf{A} \in T$ and $\mathsf{A}, \% \rightarrow \mathsf{B} \in T$, we can conclude $w = 0^n = 0^{n-1}0\% \in \Lambda_\mathsf{B}$.

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Because there are no transitions from B back to A, we could first prove that, for all $w \in \{0,1\}^*$,

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In the case of N, we could use mathematical induction instead of strong string induction:

(A) for all $n \in \mathbb{N}$, $0^n \in \Lambda_{\mathsf{A}}$, and

(B) for all $n, m \in \mathbb{N}$, $0^n(11)^m \in \Lambda_{\mathsf{B}}$ (do induction on m, fixing n).

Proving that Everything Accepted is Wanted

It's tempting to try to prove that everything accepted by a finite automaton is wanted using strong string induction, with implications like

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Unfortunately, this doesn't work when a finite automaton contains %-transitions.

Instead, we do such proofs using a new induction principle that we call induction on Λ.

Principle of Induction on Λ

Theorem 3.8.4 (Principle of Induction on Λ)
Suppose M *is a finite automaton, and* $P_q(w)$ *is a property of a* $w \in \Lambda_{M,q}$*, for all* $q \in Q_M$*.*
If

then

$$\textit{for all } q \in Q_M, \textit{ for all } w \in \Lambda_{M,q}, P_q(w).$$

We refer to (†) as the inductive hypothesis.

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- $P_{s_M}(\%)$ *and*
- *for all* $q, r \in Q_M$, $x \in \textbf{Str}$ *and* $w \in \Lambda_{M,q}$,
  if $q, x \to r \in T_M$ *and (†)* $P_q(w)$, *then* $P_r(wx)$,

then

$$\textit{for all } q \in Q_M, \ \textit{for all } w \in \Lambda_{M,q}, P_q(w).$$

We refer to (†) as the inductive hypothesis.

Principle of Induction on Λ

Proof. It suffices to show that, for all $lp \in \mathbf{LP}$, for all $q \in Q_M$, if lp is valid for M, $\mathbf{startState}\,lp = s_M$ and $\mathbf{endState}\,lp = q$, then $P_q(\mathbf{label}\,lp)$. We prove this by well-founded induction on the length of lp. □

Everything Accepted is Wanted in Example

In the case of our example FA, M, we can let $P_{\mathsf{A}}(w)$ and $P_{\mathsf{B}}(w)$ be $w \in X$ and $w \in Y$, respectively, where, as before,

$$X = \{\, w \in \{0,1\}^* \mid w \text{ has an even number of } 0\text{'s} \,\},$$
$$Y = \{\, w \in \{0,1\}^* \mid w \text{ has an odd number of } 0\text{'s} \,\}.$$

Everything Accepted is Wanted in Example

Then the principle of induction on Λ tells us that

(A) for all $w \in \Lambda_{\mathsf{A}}$, $w \in X$, and

(B) for all $w \in \Lambda_{\mathsf{B}}$, $w \in Y$,

follows from showing

- **(empty string)** $\% \in X$;
- **(**$\mathsf{A}, 0 \to \mathsf{B}$**)** for all $w \in \Lambda_{\mathsf{A}}$, if (†) $w \in X$, then $w0 \in Y$;
- **(**$\mathsf{A}, 1 \to \mathsf{A}$**)** for all $w \in \Lambda_{\mathsf{A}}$, if (†) $w \in X$, then $w1 \in X$;
- **(**$\mathsf{B}, 0 \to \mathsf{A}$**)** for all $w \in \Lambda_{\mathsf{B}}$, if (†) $w \in Y$, then $w0 \in X$;
- **(**$\mathsf{B}, 1 \to \mathsf{B}$**)** for all $w \in \Lambda_{\mathsf{B}}$, if (†) $w \in Y$, then $w1 \in Y$.

We refer to (†) as the inductive hypothesis.

Everything Accepted is Wanted in Example

There are five steps to show.

- **(empty string)** Because $\% \in \{0, 1\}^*$ and $\%$ has no 0's, we have that $\% \in X$.

Everything Accepted is Wanted in Example

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- **(empty string)** Because $\% \in \{0,1\}^*$ and $\%$ has no 0's, we have that $\% \in X$.
- **($\mathsf{A}, 0 \to \mathsf{B}$)** Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w \in X$. Hence $w \in \{0,1\}^*$ and w has an even number of 0's. Thus $w0 \in \{0,1\}^*$ and $w0$ has an odd number of 0's, so that $w0 \in Y$.

Everything Accepted is Wanted in Example

There are five steps to show.

- **(empty string)** Because $\% \in \{0,1\}^*$ and $\%$ has no 0's, we have that $\% \in X$.
- **($\mathsf{A}, 0 \to \mathsf{B}$)** Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w \in X$. Hence $w \in \{0,1\}^*$ and w has an even number of 0's. Thus $w0 \in \{0,1\}^*$ and $w0$ has an odd number of 0's, so that $w0 \in Y$.
- **($\mathsf{A}, 1 \to \mathsf{A}$)** Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w \in X$. Then $w1 \in X$.

Everything Accepted is Wanted in Example

There are five steps to show.

- **(empty string)** Because $\% \in \{0,1\}^*$ and $\%$ has no 0's, we have that $\% \in X$.
- **($\mathsf{A}, 0 \rightarrow \mathsf{B}$)** Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w \in X$. Hence $w \in \{0,1\}^*$ and w has an even number of 0's. Thus $w0 \in \{0,1\}^*$ and $w0$ has an odd number of 0's, so that $w0 \in Y$.
- **($\mathsf{A}, 1 \rightarrow \mathsf{A}$)** Suppose $w \in \Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w \in X$. Then $w1 \in X$.
- **($\mathsf{B}, 0 \rightarrow \mathsf{A}$)** Suppose $w \in \Lambda_{\mathsf{B}}$, and assume the inductive hypothesis: $w \in Y$. Then $w0 \in X$.

Everything Accepted is Wanted in Example

There are five steps to show.

- **(empty string)** Because $\%\in\{0,1\}^*$ and $\%$ has no 0's, we have that $\%\in X$.
- **(**$\mathsf{A},0\to\mathsf{B}$**)** Suppose $w\in\Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w\in X$. Hence $w\in\{0,1\}^*$ and w has an even number of 0's. Thus $w0\in\{0,1\}^*$ and $w0$ has an odd number of 0's, so that $w0\in Y$.
- **(**$\mathsf{A},1\to\mathsf{A}$**)** Suppose $w\in\Lambda_{\mathsf{A}}$, and assume the inductive hypothesis: $w\in X$. Then $w1\in X$.
- **(**$\mathsf{B},0\to\mathsf{A}$**)** Suppose $w\in\Lambda_{\mathsf{B}}$, and assume the inductive hypothesis: $w\in Y$. Then $w0\in X$.
- **(**$\mathsf{B},1\to\mathsf{B}$**)** Suppose $w\in\Lambda_{\mathsf{B}}$, and assume the inductive hypothesis: $w\in Y$. Then $w1\in Y$.

Everything Accepted is Wanted in Example

Because of

(A) for all $w \in \Lambda_{\mathsf{A}}$, $w \in X$, and

(B) for all $w \in \Lambda_{\mathsf{B}}$, $w \in Y$,

we have that $\Lambda_{\mathsf{A}} \subseteq X$ and $\Lambda_{\mathsf{B}} \subseteq Y$.

Everything Accepted is Wanted in Example

Because of

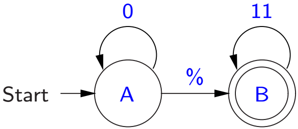
(A) for all $w \in \Lambda_{\mathsf{A}}$, $w \in X$, and

(B) for all $w \in \Lambda_{\mathsf{B}}$, $w \in Y$,

we have that $\Lambda_{\mathsf{A}} \subseteq X$ and $\Lambda_{\mathsf{B}} \subseteq Y$. Because $X \subseteq \Lambda_{\mathsf{A}}$ and $Y \subseteq \Lambda_{\mathsf{B}}$, we can conclude that $L(M) = \Lambda_{\mathsf{A}} = X$ and $\Lambda_{\mathsf{B}} = Y$.

Everything Accepted is Wanted in Second Example

Consider our second example, N, again:



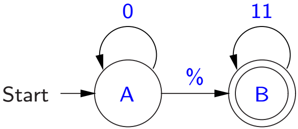
We can use induction on Λ to prove that

(A) for all $w \in \Lambda_{\mathsf{A}}$, $w \in \{0\}^*$; and

(B) for all $w \in \Lambda_{\mathsf{B}}$, $w \in \{0\}^*\{11\}^*$.

Everything Accepted is Wanted in Second Example

Consider our second example, N, again:



We can use induction on Λ to prove that

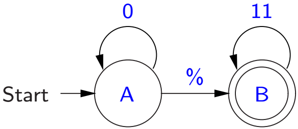
(A) for all $w \in \Lambda_{\mathsf{A}}$, $w \in \{0\}^*$; and

(B) for all $w \in \Lambda_{\mathsf{B}}$, $w \in \{0\}^*\{11\}^*$.

Thus $\Lambda_{\mathsf{A}} \subseteq \{0\}^*$ and $\Lambda_{\mathsf{B}} \subseteq \{0\}^*\{11\}^*$.

Everything Accepted is Wanted in Second Example

Consider our second example, N, again:



We can use induction on Λ to prove that

(A) for all $w \in \Lambda_{\mathsf{A}}$, $w \in \{0\}^*$; and

(B) for all $w \in \Lambda_{\mathsf{B}}$, $w \in \{0\}^*\{11\}^*$.

Thus $\Lambda_{\mathsf{A}} \subseteq \{0\}^*$ and $\Lambda_{\mathsf{B}} \subseteq \{0\}^*\{11\}^*$. Because $\{0\}^* \subseteq \Lambda_{\mathsf{A}}$ and $\{0\}^*\{11\}^* \subseteq \Lambda_{\mathsf{B}}$, we can conclude that $\Lambda_{\mathsf{A}} = \{0\}^*$ and $L(N) = \Lambda_{\mathsf{B}} = \{0\}^*\{11\}^*$.