

## *Chapter 4: Context-free Languages*

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On the other hand, the context-free languages have weaker closure properties than the regular languages, and we won't be able to give algorithms for checking grammar equivalence or minimizing the size of grammars.

## *4.1: Grammars, Parse Trees and Context-free Languages*

In this section, we:

- say what (context-free) grammars are;
- use the notion of a parse tree to say what grammars mean;
- say what it means for a language to be context-free; and
- begin to show how grammars can be processed using Forlan.

## (Context-free) Grammars

A *context-free grammar* (or just *grammar*)  $G$  consists of:

- a finite set  $Q_G$  of symbols (we call the elements of  $Q_G$  the *variables* of  $G$ );
- an element  $s_G$  of  $Q_G$  (we call  $s_G$  the *start variable* of  $G$ ); and
- a finite subset  $P_G$  of  $\{(q, x) \mid q \in Q_G \text{ and } x \in \mathbf{Str}\}$  (we call the elements of  $P_G$  the *productions* of  $G$ , and we often write  $(q, x)$  as  $q \rightarrow x$ ).

In a context where we are only referring to a single grammar,  $G$ , we sometimes abbreviate  $Q_G$ ,  $s_G$  and  $P_G$  to  $Q$ ,  $s$  and  $P$ , respectively.

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We write **Gram** for the set of all grammars. Since every grammar can be described by a finite sequence of ASCII characters, we have that **Gram** is countably infinite.

## Example Grammar

As an example, we can define a grammar  $G$  (of arithmetic expressions) as follows:

- $Q_G = \{E\}$ ;
- $s_G = E$ ;
- $P_G =$   
 $\{E \rightarrow E\langle\text{plus}\rangle E, E \rightarrow E\langle\text{times}\rangle E, E \rightarrow \langle\text{openPar}\rangle E\langle\text{closPar}\rangle,$   
 $E \rightarrow \langle\text{id}\rangle\}$ .

E.g., we can read the production  $E \rightarrow E\langle\text{plus}\rangle E$  as “an expression can consist of an expression, followed by a  $\langle\text{plus}\rangle$  symbol, followed by an expression”.

## *Notation for Grammars*

We typically describe a grammar by listing its productions, and grouping productions with identical left-sides into production families. Unless we say otherwise, the grammar's variables are the left-sides of all of its productions, and its start variable is the left-side of its first production.

Thus, our grammar  $G$  is

$$E \rightarrow E\langle\text{plus}\rangle E,$$

$$E \rightarrow E\langle\text{times}\rangle E,$$

$$E \rightarrow \langle\text{openPar}\rangle E\langle\text{closPar}\rangle,$$

$$E \rightarrow \langle\text{id}\rangle,$$

or

$$E \rightarrow E\langle\text{plus}\rangle E \mid E\langle\text{times}\rangle E \mid \langle\text{openPar}\rangle E\langle\text{closPar}\rangle \mid \langle\text{id}\rangle.$$



## *Forlan Syntax for Grammars*

The Forlan syntax for grammars is very similar to the notation of the preceding slide. E.g., here is how our example grammar can be described in Forlan's syntax:

```
{variables} E {start variable} E
{productions}
E -> E<plus>E; E -> E<times>E; E -> <openPar>E<closPar>;
E -> <id>
```

or

```
{variables} E {start variable} E
{productions}
E -> E<plus>E | E<times>E | <openPar>E<closPar> | <id>
```

## *Processing Grammars in Forlan*

The Forlan module `Gram` defines an abstract type `gram` (in the top-level environment) of grammars as well as a number of functions and constants for processing grammars, including:

```
val input      : string -> gram
val output    : string * gram -> unit
val numVariables : gram -> int
val numProductions : gram -> int
val equal     : gram * gram -> bool
```

During printing, Forlan merges productions into production families whenever possible.

## More on Grammars

The *alphabet* of a grammar  $G$  (**alphabet**  $G$ ) is

$$\{ a \in \mathbf{Sym} \mid \text{there are } q, x \text{ such that } q \rightarrow x \in P_G \text{ and} \\ a \in \mathbf{alphabet } x \} \\ - Q_G.$$

I.e., **alphabet**  $G$  is all of the symbols appearing in the strings of  $G$ 's productions that aren't variables.

For example, the alphabet of our example grammar  $G$  is  $\{\langle \text{plus} \rangle, \langle \text{times} \rangle, \langle \text{openPar} \rangle, \langle \text{closPar} \rangle, \langle \text{id} \rangle\}$ .

## *Grammar Alphabets in Forlan*

The Forlan module `Gram` defines a function

```
val alphabet : gram -> sym set
```

for calculating the alphabet of a grammar.

E.g., if `gram` of type `gram` is bound to our example grammar `G`, then Forlan will behave as follows:

```
- val bs = Gram.alphabet gram;  
val bs = - : sym set  
- SymSet.output("", bs);  
<id>, <plus>, <times>, <closPar>, <openPar>  
val it = () : unit
```

## *Parse Trees and Grammar Meaning*

We will explain when strings are generated by grammars using the notion of a parse tree. The set **PT** of *parse trees* is the least subset of **Tree(Sym  $\cup$  { $\%$ })** (the set of all **(Sym  $\cup$  { $\%$ })**-trees; see Section 1.3) such that:

(1) for all  $a \in \mathbf{Sym}$  and  $pts \in \mathbf{List PT}$ ,  $(a, pts) \in \mathbf{PT}$ ; and

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Note that the **(Sym  $\cup$  { $\%$ })**-tree  $\% = (%, [])$  is *not* a parse tree.

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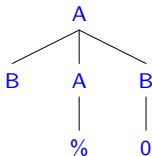
- (1) for all  $a \in \mathbf{Sym}$  and  $pts \in \mathbf{List PT}$ ,  $(a, pts) \in \mathbf{PT}$ ; and
- (2) for all  $a \in \mathbf{Sym}$ ,  $(a, [(\%, [])]) = a(\%) \in \mathbf{PT}$ .

Note that the **(Sym  $\cup$  { $\%$ })**-tree  $\% = (\%, [])$  is *not* a parse tree. It is easy to see that **PT** is countably infinite.

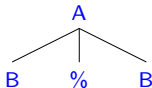


## Parse Tree Examples

For example,  $A(B, A(\%), B(0))$ , i.e.,



is a parse tree. On the other hand, although  $A(B, \%, B)$ , i.e.,



is a  $(\text{Sym} \cup \{\%\})$ -tree, it's not a parse tree, since it can't be formed using rules (1) and (2).

## *Principle of Induction on Parse Trees*

Since the set **PT** of parse trees is defined inductively, it gives rise to an induction principle.

### **Theorem 4.1.1 (Principle of Induction on Parse Trees)**

Suppose  $P(pt)$  is a property of an element  $pt \in \mathbf{PT}$ .

If

- (1) for all  $a \in \mathbf{Sym}$  and  $trs \in \mathbf{List PT}$ , if  $(\dagger)$  for all  $i \in [1 : |trs|]$ ,  $P(trs\ i)$ , then  $P((a, trs))$ , and

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for all  $pt \in \mathbf{PT}$ ,  $P(pt)$ .

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## The Yield of a Parse Tree

We define the yield of a parse tree, as follows. The function  $\mathbf{yield} \in \mathbf{PT} \rightarrow \mathbf{Str}$  is defined by structural recursion:

- for all  $a \in \mathbf{Sym}$ ,  $\mathbf{yield} a =$
- for all  $q \in \mathbf{Sym}$ ,  $n \in \mathbb{N} - \{0\}$  and  $pt_1, \dots, pt_n \in \mathbf{PT}$ ,  
 $\mathbf{yield}(q(pt_1, \dots, pt_n)) =$
- for all  $q \in \mathbf{Sym}$ ,  $\mathbf{yield}(q(\%)) =$

We say that  $w$  is the *yield of*  $pt$  iff  $w = \mathbf{yield} pt$ .

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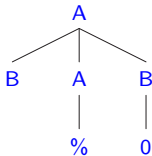
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For example, the yield of

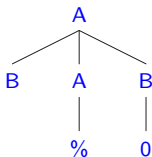


is



## *Yield Example*

For example, the yield of



is

**yield B yield(A(%)) yield(B(0)) = B % yield 0 = B%0 = B0.**

## Validity of Parse Trees for Grammars

We say when a parse tree is valid for a grammar  $G$  as follows. Define a function  $\mathbf{valid}_G \in \mathbf{PT} \rightarrow \mathbf{Bool}$  by structural recursion:

- for all  $a \in \mathbf{Sym}$ ,  $\mathbf{valid}_G a =$
- for all  $q \in \mathbf{Sym}$ ,  $n \in \mathbb{N} - \{0\}$  and  $pt_1, \dots, pt_n \in \mathbf{PT}$ ,

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We say that  $pt$  is *valid for  $G$*  iff  $\mathbf{valid}_G pt = \mathbf{true}$ . We often abbreviate  $\mathbf{valid}_G$  to  $\mathbf{valid}$ .

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## *Parse Tree Validity Examples*

Suppose  $G$  is the grammar

$$A \rightarrow BAB \mid \%,$$

$$B \rightarrow 0$$

(by convention, its variables are  $A$  and  $B$  and its start variable is  $A$ ). Let's see why the parse tree  $A(B, A(\%), B(0))$  is valid for  $G$ .

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- Since  $A \rightarrow BAB \in P_G$  and the concatenation of the root labels of the children  $B$ ,  $A(\%)$  and  $B(0)$  is  $BAB$ , the overall tree will be valid for  $G$  if these children are valid for  $G$ .



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- Since  $A \rightarrow \% \in P_G$ , the parse tree  $A(\%)$  is valid for  $G$ .

## *Validity Examples*

- Since  $B \rightarrow 0 \in P_G$  and the root label of the child  $0$  is  $0$ , the parse tree  $B(0)$  will be valid for  $G$  if the child  $0$  is valid for  $G$ .

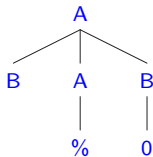
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- The child  $0$  is valid for  $G$  since  $0 \in \mathbf{alphabet } G$ .

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Thus, we have that



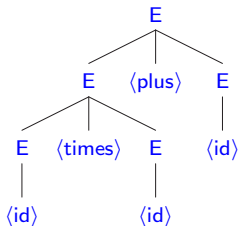
is valid for  $G$ .

## Validity Examples

Suppose  $G$  is our grammar of arithmetic expressions

$$E \rightarrow E\langle\text{plus}\rangle E \mid E\langle\text{times}\rangle E \mid \langle\text{openPar}\rangle E \langle\text{closPar}\rangle \mid \langle\text{id}\rangle.$$

Then the parse tree



is valid for  $G$ .

## The Meaning of Grammars

Suppose  $G$  is a grammar,  $w \in \mathbf{Str}$  and  $a \in \mathbf{Sym}$ . We say that  $w$  is *parsable from  $a$  using  $G$*  iff there is a parse tree  $pt$  such that:

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We say that a string  $w$  is *generated from a variable  $q \in Q_G$  using  $G$*  iff  $w \in (\mathbf{alphabet } G)^*$  and  $w$  is parsable from  $q$ .



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- the yield of  $pt$  is  $w$ .

Thus we will have that  $w \in (Q_G \cup \mathbf{alphabet } G)^*$ , and either  $a \in Q_G$  or  $[a] = w$ .

We say that a string  $w$  is *generated from a variable  $q \in Q_G$  using  $G$*  iff  $w \in (\mathbf{alphabet } G)^*$  and  $w$  is parsable from  $q$ .

And, we say that a string  $w$  is *generated by a grammar  $G$*  iff  $w$  is generated from  $s_G$  using  $G$ .

## *The Meaning of Grammars*

The *language generated by* a grammar  $G$  ( $L(G)$ ) is

$$\{ w \in \mathbf{Str} \mid w \text{ is generated by } G \}.$$

### **Proposition 4.1.3**

For all grammars  $G$ ,  $\mathbf{alphabet}(L(G)) \subseteq \mathbf{alphabet } G$ .

## *Grammar Meaning Examples*

Let  $G$  be the example grammar

$$A \rightarrow BAB \mid \%,$$

$$B \rightarrow 0.$$

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is valid for  $G$ , has  $s_G = A$  as its root label, and has  $00$  as its yield.

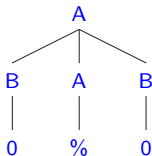
## Grammar Meaning Examples

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## Grammar Meaning Examples

Suppose  $G$  is our grammar of arithmetic expressions

$$E \rightarrow E\langle\text{plus}\rangle E \mid E\langle\text{times}\rangle E \mid \langle\text{openPar}\rangle E \langle\text{closPar}\rangle \mid \langle\text{id}\rangle.$$

Then  $\langle\text{id}\rangle\langle\text{times}\rangle\langle\text{id}\rangle\langle\text{plus}\rangle\langle\text{id}\rangle$  is generated by  $G$  since  $\langle\text{id}\rangle\langle\text{times}\rangle\langle\text{id}\rangle\langle\text{plus}\rangle\langle\text{id}\rangle \in (\mathbf{\text{alphabet } G})^*$  and the parse tree

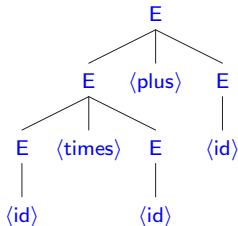
is valid for  $G$ , has  $s_G = E$  as its root label, and has  $\langle\text{id}\rangle\langle\text{times}\rangle\langle\text{id}\rangle\langle\text{plus}\rangle\langle\text{id}\rangle$  as its yield.

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## Context-free Languages

A language  $L$  is *context-free* iff  $L = L(G)$  for some  $G \in \mathbf{Gram}$ . We define

$$\begin{aligned}\mathbf{CFLan} &= \{ L(G) \mid G \in \mathbf{Gram} \} \\ &= \{ L \in \mathbf{Lan} \mid L \text{ is context-free} \}.\end{aligned}$$

Since  $\{0^0\}$ ,  $\{0^1\}$ ,  $\{0^2\}$ ,  $\dots$ , are all context-free languages, we have that  $\mathbf{CFLan}$  is infinite. But, since  $\mathbf{Gram}$  is countably infinite, it follows that  $\mathbf{CFLan}$  is also countably infinite.

Since  $\mathbf{Lan}$  is uncountable, it follows that  $\mathbf{CFLan} \subsetneq \mathbf{Lan}$ , i.e., there are non-context-free languages. Later, we will see that  $\mathbf{RegLan} \subsetneq \mathbf{CFLan}$ .



## *Equivalence of Grammars*

We say that grammars  $G$  and  $H$  are *equivalent* iff  $L(G) = L(H)$ . In other words,  $G$  and  $H$  are equivalent iff  $G$  and  $H$  generate the same language.

We define a relation  $\approx$  on **Gram** by:  $G \approx H$  iff  $G$  and  $H$  are equivalent. It is easy to see that  $\approx$  is reflexive on **Gram**, symmetric and transitive.

## *Processing Parse Trees in Forlan*

The Forlan module `PT` defines an abstract type `pt` of parse trees (in the top-level environment) along with some functions for processing parse trees:

```
val input      : string -> pt
val output     : string * pt -> unit
val height    : pt -> int
val size      : pt -> int
val equal     : pt * pt -> bool
val rootLabel : pt -> sym
val yield     : pt -> str
```

The Forlan syntax for parse trees is simply the linear syntax that we've been using in this section.

## *Graphical Editor for Parse Trees*

The Java program JForlan, can be used to view and edit parse trees. It can be invoked directly, or run via Forlan. See the Forlan website for more information.

## *More Parse Tree Processing*

The Forlan module `Gram` also defines the functions

```
val checkPT : gram -> pt -> unit
val validPT : gram -> pt -> bool
```

The function `checkPT` is used to check whether a parse tree is valid for a grammar; if the answer is “no”, it explains why not and raises an exception; otherwise it simply returns `()`.

The function `validPT` checks whether a parse tree is valid for a grammar, silently returning `true` if it is, and silently returning `false` if it isn't.

## *Forlan Examples*

Suppose the identifier `gram` of type `gram` is bound to the grammar

$$A \rightarrow BAB \mid \%,$$

$$B \rightarrow 0.$$

And, suppose that the identifier `gram'` of type `gram` is bound to our grammar of arithmetic expressions

$$E \rightarrow E\langle\text{plus}\rangle E \mid E\langle\text{times}\rangle E \mid \langle\text{openPar}\rangle E \langle\text{closPar}\rangle \mid \langle\text{id}\rangle.$$

## *Forlan Examples*

Here are some examples of how we can process parse trees using Forlan:

```
- val pt = PT.input "";
@ A(B, A(%), B(O))
@ .
val pt = - : pt
- Sym.output("", PT.rootLabel pt);
A
val it = () : unit
- Str.output("", PT.yield pt);
B0
val it = () : unit
- Gram.validPT gram pt;
val it = true : bool
```

## *Forlan Examples*

```
- val pt' = PT.input "";
@ E(E(E(<id>), <times>, E(<id>)), <plus>, E(<id>))
@ .
val pt' = - : pt
- Sym.output("", PT.rootLabel pt');
E
val it = () : unit
- Str.output("", PT.yield pt');
<id><times><id><plus><id>
val it = () : unit
- Gram.validPT gram' pt';
val it = true : bool
```

## *Forlan Examples*

```
- Gram.checkPT gram pt';  
invalid production: "E -> E<plus>E"
```

*uncaught exception Error*

```
- Gram.checkPT gram' pt;  
invalid production: "A -> BAB"
```

*uncaught exception Error*

```
- PT.input "";  
@ A(B,%,B)  
@ .  
line 1: "%" unexpected
```

*uncaught exception Error*



## Grammar Synthesis

We conclude this section with a grammar synthesis example. Suppose  $X = \{0^n 1^m 2^m 3^n \mid n, m \in \mathbb{N}\}$ . How can we find a grammar  $G$  such that  $L(G) = X$ ?

The key is to think of generating the strings of  $X$

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## Grammar Synthesis

We conclude this section with a grammar synthesis example.

Suppose  $X = \{0^n 1^m 2^m 3^n \mid n, m \in \mathbb{N}\}$ . How can we find a grammar  $G$  such that  $L(G) = X$ ?

The key is to think of generating the strings of  $X$  from the outside in, in two phases.

In the first phase, one generates pairs of 0's and 3's, and, in the second phase, one generates pairs of 1's and 2's. E.g., a string could be formed in the following stages:

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## *Grammar Synthesis*

This analysis leads us to the grammar

$$A \rightarrow$$

where **A** corresponds to the first phase, and **B** to the second phase.

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This analysis leads us to the grammar

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$$A \rightarrow B,$$

$$B \rightarrow 1B2,$$

$$B \rightarrow$$

where **A** corresponds to the first phase, and **B** to the second phase.

## *Grammar Synthesis*

This analysis leads us to the grammar

$$A \rightarrow 0A3,$$

$$A \rightarrow B,$$

$$B \rightarrow 1B2,$$

$$B \rightarrow \%,$$

where **A** corresponds to the first phase, and **B** to the second phase.

## Grammar Synthesis

For example, here is how the string **001233** may be parsed using **G**:

