

4.7: Closure Properties of Context-free Languages

In this section, we define union, concatenation and closure operations/algorithms on grammars. As a result, we will have that the context-free languages are closed under union, concatenation and closure.

In Section 4.10, we will see that the context-free languages aren't closed under intersection, complementation and set difference.

But we are able to define operations/algorithms for:

- intersecting a grammar and an empty-string finite automaton; and
- subtracting a deterministic finite automaton from a grammar.

Thus, if L_1 is a context-free language, and L_2 is a regular language, we will have that $L_1 \cap L_2$ and $L_1 - L_2$ are context-free.

The book shows several additional closure properties of context-free languages, in addition to giving the corresponding operations on grammars.

Basic Grammars and Operations on Grammars

The grammar, **emptyStr**, with variable A and production $A \rightarrow \%$ generates the language $\{\%\}$.

The grammar, **emptySet**, with variable A and no productions generates the language \emptyset .

If w is a string, then the grammar with variable A and production $A \rightarrow w$ generates the language $\{w\}$. Actually, we must be careful to choose a variable that doesn't occur in w . We can do that by adding as many nested $<$ and $>$ around A as needed (A , $\langle A \rangle$, $\langle\langle A \rangle\rangle$, etc.).

This defines functions **strToGram** \in **Str** \rightarrow **Gram** and **symToGram** \in **Sym** \rightarrow **Gram**.

Union Operation

Suppose G_1 and G_2 are grammars. We can define a grammar H such that $L(H) = L(G_1) \cup L(G_2)$ by unioning together the variables and productions of G_1 and G_2 , and adding a new start variable q , along with productions

$$q \rightarrow s_{G_1} \mid s_{G_2}.$$

What do we have to know about G_1 , G_2 and q for the above to be valid?

- $Q_{G_1} \cap Q_{G_2} = \emptyset$ and $q \notin Q_{G_1} \cup Q_{G_2}$; and
- **alphabet** $G_1 \cap Q_{G_2} = \emptyset$, **alphabet** $G_2 \cap Q_{G_1} = \emptyset$ and $q \notin$ **alphabet** $G_1 \cup$ **alphabet** G_2 .

Union Operation

Our official union operation for grammars renames the variables of G_1 and G_2 , and chooses the start variable q , in a uniform way that makes the preceding properties hold.

This gives us a function $\text{union} \in \text{Gram} \times \text{Gram} \rightarrow \text{Gram}$.

We do something similar when defining the other closure operations. In what follows, though, we'll ignore this issue, so as to keep things simple.

Concatenation and Closure Operations

Suppose G_1 and G_2 are grammars. We can define a grammar H such that $L(H) = L(G_1)L(G_2)$ by unioning together the variables and productions of G_1 and G_2 , and adding a new start variable q , along with production

$$q \rightarrow s_{G_1}s_{G_2}.$$

This gives us a function **concat** $\in \mathbf{Gram} \times \mathbf{Gram} \rightarrow \mathbf{Gram}$.

Suppose G is a grammar. We can define a grammar H such that $L(H) = L(G)^*$ by adding to the variables and productions of G a new start variable q , along with productions

$$q \rightarrow \% \mid s_G q.$$

This gives us a function **closure** $\in \mathbf{Gram} \rightarrow \mathbf{Gram}$.

Intersection of Grammar and EFA

We now consider an algorithm for intersecting a grammar G with an EFA M , resulting in **simplify** H , where the grammar H is defined as follows.

For all $p \in Q_G$ and $q, r \in Q_M$, H has a variable $\langle p, q, r \rangle$ that generates

$$\{ w \in (\mathbf{alphabet} \ G)^* \mid w \in \Pi_{G,p} \text{ and } r \in \Delta_M(\{q\}, w) \}.$$

The remaining variable of H is A , which is its start variable.

For each $r \in A_M$, H has a production

$$A \rightarrow \langle s_G, s_M, r \rangle.$$

For each $\%$ -production $p \rightarrow \%$ of G and $q, r \in Q_M$, if $r \in \Delta_M(\{q\}, \%)$, then H will have the production

$$\langle p, q, r \rangle \rightarrow \%.$$

Intersection of Grammar and EFA

To say what the remaining productions of H are, define a function

$$f \in (\mathbf{alphabet} \ G \cup Q_G) \times Q_M \times Q_M \rightarrow \mathbf{alphabet} \ G \cup Q_H$$

by: for all $a \in \mathbf{alphabet} \ G \cup Q_G$ and $q, r \in Q_M$,

$$f(a, q, r) = \begin{cases} a, & \text{if } a \in \mathbf{alphabet} \ G, \text{ and} \\ \langle a, q, r \rangle, & \text{if } a \in Q_G. \end{cases}$$

For all $p \in Q_G$, $n \in \mathbb{N} - \{0\}$, $a_1, \dots, a_n \in \mathbf{Sym}$ and $q_1, \dots, q_{n+1} \in Q_M$, if

- $p \rightarrow a_1 \cdots a_n \in P_G$, and
- for all $i \in [1 : n]$, if $a_i \in \mathbf{alphabet} \ G$, then $q_{i+1} \in \Delta_M(\{q_i\}, a_i)$,

then we let

$$\langle p, q_1, q_{n+1} \rangle \rightarrow f(a_1, q_1, q_2) \cdots f(a_n, q_n, q_{n+1})$$

be production of H .

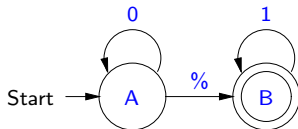
Intersection of Grammar and EFA

This gives us a function $\text{inter} \in \mathbf{Gram} \times \mathbf{EFA} \rightarrow \mathbf{Gram}$.

For example, let G be the grammar

$$A \rightarrow \% \mid 0A1A \mid 1A0A,$$

and M be the EFA



so that G generates all elements of $\{0,1\}^*$ with an equal number of 0's and 1's, and M accepts $\{0\}^*\{1\}^*$.

Intersection of Grammar and EFA

Then **simplify H** is

$$\begin{aligned}A &\rightarrow \langle A, A, B \rangle, \\ \langle A, A, A \rangle &\rightarrow \%, \\ \langle A, A, B \rangle &\rightarrow \%, \\ \langle A, A, B \rangle &\rightarrow 0\langle A, A, A \rangle 1\langle A, B, B \rangle, \\ \langle A, A, B \rangle &\rightarrow 0\langle A, A, B \rangle 1\langle A, B, B \rangle, \\ \langle A, A, B \rangle &\rightarrow 0\langle A, B, B \rangle 1\langle A, B, B \rangle, \\ \langle A, B, B \rangle &\rightarrow \%.\end{aligned}$$

Note that simplification eliminated the variable $\langle A, B, A \rangle$. If we hand simplify further, we can turn this into:

$$\begin{aligned}A &\rightarrow \langle A, A, B \rangle, \\ \langle A, A, B \rangle &\rightarrow \% \mid 0\langle A, A, B \rangle 1\end{aligned}$$

Intersection of Grammar and EFA

To prove that our intersection algorithm is correct, we'll need two lemmas.

Lemma 4.7.1

For $p \in Q_G$, let the property $P_p(w)$, for $w \in \Pi_{G,p}$, be:

for all $q, r \in Q_M$, if $r \in \Delta_M(\{q\}, w)$, then $w \in \Pi_{H, \langle p, q, r \rangle}$.

Then, for all $p \in Q_G$, for all $w \in \Pi_{G,p}$, $P_p(w)$.

Proof. By induction on Π . We use the fact that, if $n \in \mathbb{N} - \{0\}$, $q_1, q_{n+1} \in Q_M$, $w_1, \dots, w_n \in \mathbf{Str}$ and $q_{n+1} \in \Delta_M(\{q_1\}, w_1 \cdots w_n)$, then there are $q_2, \dots, q_n \in Q_M$ such that $q_{i+1} \in \Delta_M(\{q_i\}, w_i)$, for all $i \in [1 : n]$. (This is true because M is an EFA; if M were an FA, we wouldn't be able to conclude this.) \square

Intersection of Grammar and EFA

Lemma 4.7.2

Let the property $P_A(w)$, for $w \in \Pi_{H,A}$, be

$$w \in L(G) \text{ and } w \in L(M).$$

For $p \in Q_G$ and $q, r \in Q_M$, let the property $P_{\langle p,q,r \rangle}(w)$, for $w \in \Pi_{H,\langle p,q,r \rangle}$, be

$$w \in \Pi_{G,p} \text{ and } r \in \Delta_M(\{q\}, w).$$

Then:

- (1) For all $w \in \Pi_{H,A}$, $P_A(w)$.
- (2) For all $p \in Q_G$ and $q, r \in Q_M$, for all $w \in \Pi_{H,\langle p,q,r \rangle}$, $P_{\langle p,q,r \rangle}(w)$.

Proof. We proceed by induction on Π . \square

Intersection of Grammar and EFA

Lemma 4.7.3

$$L(H) = L(G) \cap L(M).$$

Proof. $L(H) \subseteq L(G) \cap L(M)$ follows by Lemma 4.7.2(1).

For the other inclusion, suppose $w \in L(G) \cap L(M)$, so that $w \in \Pi_{G, s_G}$ and $r \in \Delta_M(\{s_M\}, w)$, for some $r \in A_M$. By Lemma 4.7.1, it follows that $w \in \Pi_{H, \langle s_G, s_M, r \rangle}$. But because $r \in A_M$, we have that $A \rightarrow \langle s_G, s_M, r \rangle$ is a production of H . Thus $w \in \Pi_{H, A} = L(H)$. \square

Difference of Grammar and DFA

Given a grammar G and a DFA M , we can define the difference of G and M to be

$\text{inter}(G, \text{complement}(M, \text{alphabet } G))$.

This is analogous to what we did when defining the difference of DFAs.

This gives us a function **$\text{minus} \in \text{Gram} \times \text{DFA} \rightarrow \text{Gram}$.**

Summary of Closure Properties

Theorem 4.7.4

Suppose $L, L_1, L_2 \in \mathbf{CFLan}$ and $L' \in \mathbf{RegLan}$. Then:

- (1) $L_1 \cup L_2 \in \mathbf{CFLan}$;
- (2) $L_1 L_2 \in \mathbf{CFLan}$;
- (3) $L^* \in \mathbf{CFLan}$;
- (4) $L \cap L' \in \mathbf{CFLan}$; and
- (5) $L - L' \in \mathbf{CFLan}$.

Operations on Grammars in Forlan

The Forlan module `Gram` defines the following constants and operations on grammars:

```
val emptyStr    : gram
val emptySet    : gram
val fromStr     : str -> gram
val fromSym     : sym -> gram
val union       : gram * gram -> gram
val concat      : gram * gram -> gram
val closure     : gram -> gram
val fromStrSet  : str set -> gram
val inter       : gram * efa -> gram
val minus       : gram * dfa -> gram
```

The functions `fromStr` and `fromSym` are also available in the top-level environment with the names

```
val strToGram  : str -> gram
val symToGram  : sym -> gram
```

Forlan Examples

For example, we can construct a grammar G such that $L(G) = \{01\} \cup \{10\}\{11\}^*$, as follows.

```
- val gram1 = strToGram(Str.fromString "01");  
val gram1 = - : gram  
- val gram2 = strToGram(Str.fromString "10");  
val gram2 = - : gram  
- val gram3 = strToGram(Str.fromString "11");  
val gram3 = - : gram  
- val gram =  
= Gram.union(gram1,  
=           Gram.concat(gram2,  
=                   Gram.closure gram3));  
val gram = - : gram
```


Forlan Examples

```
- val gram' = Gram.renameVariablesCanonically gram;  
val gram' = - : gram  
- Gram.output("", gram');  
{variables} A, B, C, D, E, F {start variable} A  
{productions}  
A -> B | C; B -> 01; C -> DE; D -> 10; E -> % | FE;  
F -> 11  
val it = () : unit
```

Forlan Examples

We can use `Gram.fromStrSet` as follows:

```
- val gram'' =  
= Gram.fromStrSet  
= (StrSet.fromString "0, 01, 010, 0101");  
val gram'' = - : gram  
- val gram''' = Gram.renameVariablesCanonically  
gram'';  
val gram''' = - : gram  
- Gram.output("", gram''');  
{variables} A, B, C, D, E {start variable} A  
{productions}  
A -> B | C | D | E; B -> 0; C -> 01; D -> 010;  
E -> 0101  
val it = () : unit
```

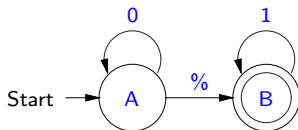
Forlan Examples

And here are examples of how we can use `Gram.inter` and `Gram.minus`.

Let `gram` be the grammar

$$A \rightarrow \% \mid 0A1A \mid 1A0A,$$

and `efa` be the EFA



Forlan Examples

```
- val gram' = Gram.inter(gram, efa);
val gram' = - : gram
- Gram.output("", gram');
{variables} A, <A,A,A>, <A,A,B>, <A,B,B>
{start variable} A
{productions}
A -> <A,A,B>; <A,A,A> -> %;
<A,A,B> ->
% | 0<A,A,A>1<A,B,B> | 0<A,A,B>1<A,B,B> |
0<A,B,B>1<A,B,B>;
<A,B,B> -> %
val it = () : unit
- fun elimVars(gram, nil)      = gram
=   | elimVars(gram, q :: qs) =
=       elimVars(Gram.eliminateVariable(gram, q), qs);
val elimVars = fn : gram * sym list -> gram
```

Forlan Examples

```
- val gram'' =
=     elimVars
=     (gram',
=     [Sym.fromString "<A,A,A>",
=     Sym.fromString "<A,B,B>"]);
val gram'' = - : gram
- val gram''' =
=     Gram.renameVariablesCanonically
=     (Gram.restart(Gram.simplify gram''));
val gram''' = - : gram
- Gram.output("", gram''');
{variables} A {start variable} A
{productions} A -> % | OA1
val it = () : unit
```

Forlan Examples

```
- val dfa =  
=      DFA.renameStatesCanonically  
=      (DFA.minimize(nfaToDFA(efaToNFA efa)));  
val dfa = - : dfa  
- val gram'' = Gram.minus(gram, dfa);  
val gram'' = - : gram  
- Gram.generated gram'' (Str.fromString "0101");  
val it = true : bool  
- Gram.generated gram'' (Str.fromString "0011");  
val it = false : bool
```

Grammar Synthesis Example

We'll end this section with a more sophisticated example. Define a language X by:

$$X = \{ 0^i 1^j 2^k 3^l \mid i, j, k, l \in \mathbb{N} \text{ and } i < l \text{ and } j > k \text{ and } i + j \text{ is even and } k + l \text{ is odd} \}.$$

Is X context-free?

Yes. Let's see how Forlan can help us come up with a grammar generating X .

Grammar Synthesis Example

First, we put the text

```
{variables} A, B, <1>, <3> {start variable} A
{productions}
A -> 0A3 | B<3>;
B -> 1B2 | <1>;
<1> -> 1 | 1<1>;
<3> -> 3 | 3<3>
```

for a grammar generating $\{0^i1^j2^k3^l \mid i, j, k, l \in \mathbb{N} \text{ and } i < l \text{ and } j > k\}$ in the file `seq0123-where-0lt3-and-1gt2-gram.txt`.

Grammar Synthesis Example

Next we put the text

```
{states} A, B {start state} A {accepting states} A
{transitions}
A, 0 -> B; A, 1 -> B; A, 2 -> A; A, 3 -> A;
B, 0 -> A; B, 1 -> A; B, 2 -> B; B, 3 -> B
```

for a DFA accepting all elements of $\{0,1,2,3\}^*$ in which the sum of the numbers of 0's and 1's is even in the file `even0plus1-alf23-dfa.txt`.

Grammar Synthesis Example

Then we load the grammar and DFA into Forlan:

```
- val seq0123Where0lt3And1gt2Gram =  
=      Gram.input  
=      "seq0123-where-0lt3-and-1gt2-gram.txt";  
val seq0123Where0lt3And1gt2Gram = - : gram  
- val even0plus1Alp23DFA =  
=      DFA.input "even0plus1-alp23-dfa.txt";  
val even0plus1Alp23DFA = - : dfa
```

Grammar Synthesis Example

Next we carry out some standard definitions:

```
- val regToDFA =  
=       nfaToDFA o efaToNFA o faToEFA o regToFA;  
val regToDFA = fn : reg -> dfa  
- val minAndRen =  
=       DFA.renameStatesCanonically o DFA.minimize;  
val minAndRen = fn : dfa -> dfa  
- val syms0123 = SymSet.fromString "0, 1, 2, 3";  
val syms0123 = - : sym set  
- val allStrReg = Reg.closure(Reg.allSym syms0123);  
val allStrReg = - : reg  
- val allStrDFA = minAndRen(regToDFA allStrReg);  
val allStrDFA = - : dfa
```

Grammar Synthesis Example

Next we convert `even0plus1Alp23DFA` into a DFA `odd2plus3Alp01DFA` that accepts all elements of $\{0, 1, 2, 3\}^*$ in which the sum of the numbers of 2's and 3's is odd:

```
- val swap23for01 =  
=     SymRel.fromString  
=     "(0, 2), (1, 3), (2, 0), (3, 1)";  
val swap23for01 = - : sym_rel  
- val odd2plus3Alp01DFA =  
=     minAndRen  
=     (DFA.minus  
=     (allStrDFA,  
=     DFA.renameAlphabet  
=     (even0plus1Alp23DFA, swap23for01)));  
val odd2plus3Alp01DFA = - : dfa
```

See Section 3.12 of the book for discussion of `DFA.renameAlphabet`.

Grammar Synthesis Example

Next we create a DFA `even0plus1AndOdd2plus3DFA` accepting all elements of $\{0, 1, 2, 3\}^*$ in which the sum of the numbers of 0's and 1's is even, and the sum of the numbers of 2's and 3's is odd:

```
- val even0plus1AndOdd2plus3DFA =  
=     minAndRen  
=     (DFA.inter  
=     (even0plus1Alp23DFA, odd2plus3Alp01DFA));  
val even0plus1AndOdd2plus3DFA = - : dfa
```

Grammar Synthesis Example

And then we create our first grammar, `gram0`, generating X :

```
- val gram0 =  
=       Gram.renameVariablesCanonically  
=       (Gram.inter  
=       (seq0123Where0lt3And1gt2Gram,  
=       injDFAToEFA even0plus1And0odd2plus3DFA));  
val gram0 = - : gram  
- Gram.output("", gram0);  
{variables} A, B, C, D, E, F, G, H, I, J, K, L, M  
{start variable} A  
{productions}  
A -> B; B -> DK | EM | OC3; C -> FJ | GL | OB3;  
D -> H | 1G2; E -> 1F2; F -> I | 1E2; G -> 1D2;  
H -> 1I; I -> 1 | 1H; J -> 3L; K -> 3 | 3M;  
L -> 3 | 3J; M -> 3K  
val it = () : unit
```

Grammar Synthesis Example

In the grammar `gram0`, there are opportunities for hand-simplification using Forlan:

```
- fun elimVars(gram, nil)      = gram
=   | elimVars(gram, q :: qs) =
=       elimVars
=       (Gram.eliminateVariable
=       (gram, Sym.fromString q),
=       qs);
val elimVars = fn : gram * string list -> gram
- val gram1 =
=       Gram.renameVariablesCanonically
=       (elimVars
=       (Gram.restart gram0,
=       ["E", "G", "H", "J", "M", "C"]));
val gram1 = - : gram
```

Grammar Synthesis Example

```
- Gram.output("", gram1);  
{variables} A, B, C, D, E, F {start variable} A  
{productions}  
A -> BE | 00A33 | 0C3F3 | 1C23E | 01B2F3;  
B -> 1D | 11B22; C -> D | 11C22; D -> 1 | 11D;  
E -> 3 | 33E; F -> 3 | 33F  
val it = () : unit  
- Gram.numVariables gram1;  
val it = 6 : int  
- Gram.numProductions gram1;  
val it = 15 : int
```


Grammar Synthesis Example

In `gram1`, `E` and its productions, and `F` and its productions, have the same form. There is no reason to have both of them, and so we can remove `F` and its productions, replacing all occurrences of `F` in the remaining productions by `E`. This gives us the grammar:

$A \rightarrow BE \mid 00A33 \mid 0C3E3 \mid 1C23E \mid 01B2E3$

$B \rightarrow 1D \mid 11B22$

$C \rightarrow D \mid 11C22$

$D \rightarrow 1 \mid 11D$

$E \rightarrow 3 \mid 33E.$

Grammar Synthesis Example

Because **E** generates only strings of **3**'s, we can replace the occurrences of **E3** on the right-hand sides of **A**'s productions by **3E**, yielding:

$$A \rightarrow BE \mid 00A33 \mid 0C33E \mid 1C23E \mid 01B23E$$

$$B \rightarrow 1D \mid 11B22$$

$$C \rightarrow D \mid 11C22$$

$$D \rightarrow 1 \mid 11D$$

$$E \rightarrow 3 \mid 33E.$$

Grammar Synthesis Example

Next, we note that

$$\Pi_D = \{1^m \mid m \in \mathbb{N} \text{ and } m \text{ is odd}\},$$

$$\Pi_C = \{1^{2n}1^m2^{2n} \mid n, m \in \mathbb{N} \text{ and } m \text{ is odd}\},$$

$$\begin{aligned}\Pi_B &= \{1^{2n}11^m2^{2n} \mid n, m \in \mathbb{N} \text{ and } m \text{ is odd}\} \\ &= \{11^{2n}1^m2^{2n} \mid n, m \in \mathbb{N} \text{ and } m \text{ is odd}\} \\ &= \{1\}\Pi_C.\end{aligned}$$

Thus we can remove **B** and its productions, replacing all occurrences of **B** by **1C**:

$$A \rightarrow 1CE \mid 00A33 \mid 0C33E \mid 1C23E \mid 011C23E$$

$$C \rightarrow D \mid 11C22$$

$$D \rightarrow 1 \mid 11D$$

$$E \rightarrow 3 \mid 33E.$$

Grammar Synthesis Example

Since **D** is only used in a production of **C**, we can combine the productions of **C** and **D**, yielding

$$C \rightarrow 1 \mid 11C \mid 11C22.$$

This give us our final grammar:

$$A \rightarrow 1CE \mid 00A33 \mid 0C33E \mid 1C23E \mid 011C23E$$

$$C \rightarrow 1 \mid 11C \mid 11C22$$

$$E \rightarrow 3 \mid 33E.$$

or

$$A \rightarrow 1BC \mid 1B23C \mid 0B33C \mid 011B23C \mid 00A33$$

$$B \rightarrow 1 \mid 11B \mid 11B22$$

$$C \rightarrow 3 \mid 33C.$$