

1.2: Induction

In the section, we consider several induction principles, i.e., methods for proving that every element x of some set A has some property $P(x)$.

Principle of Mathematical Induction

Theorem 1.2.1 (Principle of Mathematical Induction)

Suppose $P(n)$ is a property of a natural number n .

If

(basis step)

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for all $n \in \mathbb{N}$, if then

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for all $n \in \mathbb{N}$, $P(n)$.

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for all $n \in \mathbb{N}$, if $(\dagger) P(n)$, then $P(n + 1)$,

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for all $n \in \mathbb{N}$, $P(n)$.

We refer to the formula (\dagger) as the *inductive hypothesis*.

Principle of Strong Induction

Theorem 1.2.4 (Principle of Strong Induction)

Suppose $P(n)$ is a property of a natural number n .

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Suppose $P(n)$ is a property of a natural number n .

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for all $n \in \mathbb{N}$,

if (\dagger) for all $m \in \mathbb{N}$, if $m < n$, then $P(m)$,

then $P(n)$,

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Proof. Follows by mathematical induction, but using a property $Q(n)$ derived from $P(n)$. See the book. \square

Example Proof Using Strong Induction

Proposition 1.2.5

Every nonempty set of natural numbers has a least element.

Proof. Let X be a nonempty set of natural numbers.

We begin by using strong induction to show that, for all $n \in \mathbb{N}$,

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if $m \in X$, then X has a least element.

But $m \in X$, and thus X has a least element. This completes our strong induction.

Example Proof (Cont.)

Proof (cont.). Now we use the result of our strong induction to prove that X has a least element. Since X is a nonempty subset of \mathbb{N} , there is an $n \in \mathbb{N}$ such that $n \in X$. By the result of our induction, we can conclude that

if $n \in X$, then X has a least element.

But $n \in X$, and thus X has a least element. \square

Well-founded Induction

We can also do induction over a well-founded relation.

A relation R on a set A is *well-founded* iff every nonempty subset X of A has an R -minimal element, where an element $x \in X$ is *R -minimal in X* iff there is no $y \in X$ such that $y R x$.

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For example, in Proposition 1.2.5, we proved that the strict total ordering $<$ on \mathbb{N} is well-founded.

On the other hand, the strict total ordering $<$ on \mathbb{Z} is *not* well-founded, as \mathbb{Z} itself lacks a $<$ -minimal element.

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Of the nonempty subsets of A , we have that $\{0\}$ and $\{1\}$ have R -minimal elements. But consider A itself. Then 0 is not R -minimal in A , because

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Of the nonempty subsets of A , we have that $\{0\}$ and $\{1\}$ have R -minimal elements. But consider A itself. Then 0 is not R -minimal in A , because $1 \in A$ and $1 R 0$. And 1 is not R -minimal in A , because $0 \in A$ and $0 R 1$. Hence R is not well-founded.

Principle of Well-founded Induction

Theorem 1.2.8 (Principle of Well-founded Induction)

Suppose A is a set, R is a well-founded relation on A , and $P(x)$ is a property of an element $x \in A$.

If

for all $x \in A$,

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When $A = \mathbb{N}$ and $R = <$, this is the same as the principle of strong induction.

Proof of Well-founded Induction

Proof. Suppose A is a set, R is a well-founded relation on A , $P(x)$ is a property of an element $x \in A$, and

(†) for all $x \in A$,
if for all $y \in A$, if $y R x$, then $P(y)$,
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Suppose, toward a contradiction, that it is not the case that, for all $x \in A$, $P(x)$. Hence there is an $x \in A$ such that $P(x)$ is false. Let $X = \{x \in A \mid P(x) \text{ is false}\}$. Thus $x \in X$, showing that X is non-empty. Because R is well-founded on A , it follows that there is a $z \in X$ that is R -minimal in X , i.e., such that there is no $y \in X$ such that $y R z$.

Proof of Well-founded Induction (Cont.)

Proof (cont.). By (†), we have that

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0 has no predecessors in $\text{pred}_{\mathbb{N}}$, and, for all $n \in \mathbb{N}$, n is the only predecessor of $n + 1$ in $\text{pred}_{\mathbb{N}}$. Consequently, if a zero/non-zero case analysis is used, a proof by well-founded induction on $\text{pred}_{\mathbb{N}}$ will look like a proof by

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Well-founded Induction on Integers via Absolute Value

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If we do a well-founded induction on R , when proving $P(n)$, for $n \in \mathbb{Z}$, we can make use of $P(m)$ for any $m \in \mathbb{Z}$ whose absolute value is strictly less than the absolute value of n .

E.g., when proving $P(-10)$, we could make use of

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E.g., when proving $P(-10)$, we could make use of $P(5)$ or $P(-9)$.